Homework 1 - Linear operators and eigenspaces

Q1.1. Show that if A is a linear operator then $AA^\dagger$ is Hermitian.

A1.1. Using Eqs. (1.1.15) and (1.1.14),

$$\left(AA^\dagger\right)^\dagger = A^{\dagger\dagger}A^\dagger = AA^\dagger \tag{A1.1.1}$$

Therefore $AA^\dagger$ satisfies Eq. (1.1.17) and so is Hermitian.

Q1.2. Show that the eigenvalues α of a Hermitian operator satisfy

$$\alpha^* = \alpha \tag{Q1.2.1}$$

A1.2. Let

$$H\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle \tag{A1.2.1}$$

Then, using Eqs. (1.1.13) and (1.1.17),

$$\alpha^*\left\langle\alpha|\alpha\right\rangle = \left(\left\langle\alpha\right|H\left|\alpha\right\rangle\right)^* = \left\langle\alpha\right|H^\dagger\left|\alpha\right\rangle = \left\langle\alpha\right|H\left|\alpha\right\rangle = \alpha\left\langle\alpha|\alpha\right\rangle \tag{A1.2.2}$$

Therefore, using Eq. (1.1.10), the eigenvalues of a Hermitian operator satisfy Eq. (Q1.2.1).

Q1.3. Show that the eigenspaces of a Hermitian operator are orthogonal.

A1.3. Let

$$H\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle \tag{A1.3.1}$$
$$H\left|\beta\right\rangle = \beta\left|\beta\right\rangle \tag{A1.3.2}$$

Then, using Eqs. (Q1.2.1), (1.1.13) and (1.1.17),

$$\alpha\left\langle\alpha|\beta\right\rangle = \alpha^*\left\langle\alpha|\beta\right\rangle = \left(\left\langle\beta\right|H\left|\alpha\right\rangle\right)^* = \left\langle\alpha\right|H^\dagger\left|\beta\right\rangle = \left\langle\alpha\right|H\left|\beta\right\rangle = \beta\left\langle\alpha|\beta\right\rangle \tag{A1.3.3}$$

Therefore the eigenvectors of a Hermitian operator satisfy

$$\left\langle\alpha|\beta\right\rangle = 0 \qquad (\alpha \neq \beta) \tag{A1.3.4}$$

i.e. the eigenspaces of a Hermitian operator are orthogonal.

Q1.4. The linear operators A and B have commutator [1]

$$[A, B] = \gamma \qquad (\gamma \neq 0) \tag{Q1.4.1}$$

Show that they cannot have a common eigenvector.

[1] The right hand side of this equation is the linear operator $\gamma \times 1$, where $\gamma \in \mathbb{C}$ and 1 is the identity operator, but is just written as γ for simplicity.

A1.4. Suppose A and B have a common eigenvector

$$A\,|\alpha,\beta\rangle = \alpha\,|\alpha,\beta\rangle \tag{A1.4.1}$$

$$B\,|\alpha,\beta\rangle = \beta\,|\alpha,\beta\rangle \tag{A1.4.2}$$

Then, using Eqs. (Q1.4.1) and (1.1.16),

$$\gamma\,|\alpha,\beta\rangle = [A,B]\,|\alpha,\beta\rangle = (\alpha\beta - \beta\alpha)\,|\alpha,\beta\rangle = 0 \tag{A1.4.3}$$

which contradicts our assumptions. Therefore A and B cannot have a common eigenvector.