Homework 3 - Commutators and operations

Q3.1. The Hermitian operators $\hat{x}$ and $\hat{p}$ satisfy

$$[\hat{x}, \hat{p}] = i \tag{Q3.1.1}$$

and

$$\hat{x} \left|x\right\rangle = x \left|x\right\rangle \tag{Q3.1.2}$$

Using the Baker-Campbell-Hausdorff formula

$$e^A e^B = \exp\left(A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A, [A, B]] + \frac{1}{12}[B, [B, A]] + \dots\right) \tag{Q3.1.3}$$

calculate

$$e^{\hat{x}} \left(e^{-ia\hat{p}} \left|x\right\rangle\right) \tag{Q3.1.4}$$

and hence interpret

$$e^{-ia\hat{p}} \tag{Q3.1.5}$$

Q3.2. Let

$$\langle a|b\rangle = \begin{cases} 1 & \text{if} \quad a = b \\ 0 & \text{if} \quad a \neq b \end{cases} \tag{Q3.2.1}$$

for $a, b \in \{x, y, z\}$,

$$L_{xy} = i \left|y\right\rangle \left\langle x\right| - i \left|x\right\rangle \left\langle y\right| \tag{Q3.2.2}$$
$$L_{yz} = i \left|z\right\rangle \left\langle y\right| - i \left|y\right\rangle \left\langle z\right| \tag{Q3.2.3}$$
$$L_{zx} = i \left|x\right\rangle \left\langle z\right| - i \left|z\right\rangle \left\langle x\right| \tag{Q3.2.4}$$

and

$$L^2 = L_{xy}^2 + L_{yz}^2 + L_{zx}^2 \tag{Q3.2.5}$$

Calculate

$$[L_A, L_B] \tag{Q3.2.6}$$

and

$$\left[L_A, L^2\right] \tag{Q3.2.7}$$

for $A, B \in \{xy, yz, zx\}$.

Q3.3. Using

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{Q3.3.1}$$

calculate

$$\exp\left(-i\theta L_{xy}\right) \tag{Q3.3.2}$$

and interpret your answer.