Homework 3 - Commutators

Q3.1. The Hermitian operators $\hat{x}$ and $\hat{p}$ satisfy

$$[\hat{x}, \hat{p}] = i \tag{Q3.1.1}$$

and

$$\hat{x} |x\rangle = x |x\rangle \tag{Q3.1.2}$$

Using the Baker-Campbell-Hausdorff formula

$$e^A e^B = \exp\left(A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] + \frac{1}{12}[B,[B,A]] + \dots\right) \tag{Q3.1.3}$$

calculate

$$e^{\hat{x}} \left(e^{-ia\hat{p}} |x\rangle\right) \tag{Q3.1.4}$$

and hence interpret

$$e^{-ia\hat{p}} \tag{Q3.1.5}$$

A3.1. Using Eqs. (Q3.1.1) and (Q3.1.3),

$$e^{\hat{x}} e^{-ia\hat{p}} = e^{\hat{x} - ia\hat{p} + \frac{a}{2}} = e^a e^{-ia\hat{p}} e^{\hat{x}} \tag{A3.1.1}$$

therefore, using Eq. (Q3.1.2),

$$e^{\hat{x}} \left(e^{-ia\hat{p}} |x\rangle\right) = e^a e^{-ia\hat{p}} e^{\hat{x}} |x\rangle = e^a e^{-ia\hat{p}} e^x |x\rangle = e^{x+a} \left(e^{-ia\hat{p}} |x\rangle\right) \tag{A3.1.2}$$

Thus we can interpret $\exp(-ia\hat{p})$ as a translation in the x direction through a distance a.

Q3.2. Let

$$\langle a|b\rangle = \begin{cases} 1 & \text{if} \quad a = b \\ 0 & \text{if} \quad a \neq b \end{cases} \tag{Q3.2.1}$$

for $a, b \in \{x, y, z\}$,

$$L_{xy} = i |y\rangle \langle x| - i |x\rangle \langle y| \tag{Q3.2.2}$$

$$L_{yz} = i |z\rangle \langle y| - i |y\rangle \langle z| \tag{Q3.2.3}$$

$$L_{zx} = i |x\rangle \langle z| - i |z\rangle \langle x| \tag{Q3.2.4}$$

and

$$L^2 = L_{xy}^2 + L_{yz}^2 + L_{zx}^2 \tag{Q3.2.5}$$

Calculate

$$[L_A, L_B] \tag{Q3.2.6}$$

and

$$\left[L_A, L^2\right] \tag{Q3.2.7}$$

for $A, B \in \{xy, yz, zx\}$.

A3.2. Using Eqs. (Q3.2.2), (Q3.2.3) and (Q3.2.1),

$$\begin{aligned} L_{xy}L_{yz} &= (i\left|y\right\rangle\left\langle x\right| - i\left|x\right\rangle\left\langle y\right|)(i\left|z\right\rangle\left\langle y\right| - i\left|y\right\rangle\left\langle z\right|) && \text{(A3.2.1)} \\ &= -\left|y\right\rangle\left\langle x|z\right\rangle\left\langle y\right| + \left|y\right\rangle\left\langle x|y\right\rangle\left\langle z\right| \\ &\quad + \left|x\right\rangle\left\langle y|z\right\rangle\left\langle y\right| - \left|x\right\rangle\left\langle y|y\right\rangle\left\langle z\right| && \text{(A3.2.2)} \\ &= -\left|x\right\rangle\left\langle z\right| && \text{(A3.2.3)} \end{aligned}$$

and

$$L_{yz}L_{xy} = (L_{xy}L_{yz})^\dagger = -\left|z\right\rangle\left\langle x\right| \quad \text{(A3.2.4)}$$

therefore, using Eq. (1.1.16) and (Q3.2.4),

$$[L_{xy}, L_{yz}] = iL_{zx} \quad \text{(A3.2.5)}$$

Also, by permutation symmetry ($x \to y \to z \to x$),

$$[L_{yz}, L_{zx}] = iL_{xy} \quad \text{(A3.2.6)}$$

and

$$[L_{zx}, L_{xy}] = iL_{yz} \quad \text{(A3.2.7)}$$

Similarly

$$\begin{aligned} L_{xy}^2 &= (i\left|y\right\rangle\left\langle x\right| - i\left|x\right\rangle\left\langle y\right|)(i\left|y\right\rangle\left\langle x\right| - i\left|x\right\rangle\left\langle y\right|) && \text{(A3.2.8)} \\ &= \left|x\right\rangle\left\langle x\right| + \left|y\right\rangle\left\langle y\right| && \text{(A3.2.9)} \end{aligned}$$

Therefore, using Eq. (Q3.2.5) and permutation symmetry,

$$\begin{aligned} L^2 &= 2\left(\left|x\right\rangle\left\langle x\right| + \left|y\right\rangle\left\langle y\right| + \left|z\right\rangle\left\langle z\right|\right) && \text{(A3.2.10)} \\ &= 2 && \text{(A3.2.11)} \end{aligned}$$

where here 2 means two times the identity operator. Therefore

$$\left[L_A, L^2\right] = 0 \quad \text{(A3.2.12)}$$

for $A, B \in \{xy, yz, zx\}$.

Q3.3. Using

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{(Q3.3.1)}$$

calculate

$$\exp\left(-i\theta L_{xy}\right) \quad \text{(Q3.3.2)}$$

and interpret your answer.

A3.3. Using Eqs. (Q3.2.2) and (A3.2.9),

$$L_{xy}^3 = L_{xy} \tag{A3.3.1}$$

and

$$L_{xy}^4 = L_{xy}^2 \tag{A3.3.2}$$

Therefore, using Eq. (Q3.3.1),

$$\begin{aligned}
\exp(-i\theta L_{xy}) &= \sum_{n=0}^{\infty} \frac{i^n \theta^n}{n!} L_{xy}^n & \text{(A3.3.3)}\\
&= 1 - \sum_{n=0}^{\infty} \frac{i^{2n+1}\theta^{2n+1}}{(2n+1)!} L_{xy} + \sum_{n=1}^{\infty} \frac{i^{2n}\theta^{2n}}{(2n)!} L_{xy}^2 & \text{(A3.3.4)}\\
&= 1 - i\sin\theta L_{xy} + (\cos\theta - 1) L_{xy}^2 & \text{(A3.3.5)}\\
&= \cos\theta \, |x\rangle \langle x| - \sin\theta \, |x\rangle \langle y| \\
&\quad + \sin\theta \, |y\rangle \langle x| + \cos\theta \, |y\rangle \langle y| \\
&\quad + |z\rangle \langle z| & \text{(A3.3.6)}
\end{aligned}$$

Using Eq. (A3.3.6),

$$\begin{aligned}
\exp(-i\theta L_{xy}) |x\rangle &= \cos\theta \, |x\rangle + \sin\theta \, |y\rangle & \text{(A3.3.7)}\\
\exp(-i\theta L_{xy}) |y\rangle &= -\sin\theta \, |x\rangle + \cos\theta \, |y\rangle & \text{(A3.3.8)}\\
\exp(-i\theta L_{xy}) |z\rangle &= |z\rangle & \text{(A3.3.9)}
\end{aligned}$$

Thus we can interpret $\exp(-i\theta L_{xy})$ as a rotation in the (x, y) plane through an angle θ.