Homework 4 - Angular momentum

Q4.1. Let

$$\hat{L}_{ab} = \hat{x}_a\hat{p}_b - \hat{x}_b\hat{p}_a \tag{Q4.1.1}$$

for $a, b \in \{1, 2, 3\}$, and

$$\hat{L}^2 = \hat{L}_{12}^2 + \hat{L}_{23}^2 + \hat{L}_{31}^2 \tag{Q4.1.2}$$

Show that

$$\left[\hat{L}_{ab}, \hat{L}_{bc}\right] = i\hbar\hat{L}_{ca} \tag{Q4.1.3}$$

and

$$\left[\hat{L}_{ab}, \hat{L}^2\right] = 0 \tag{Q4.1.4}$$

for $a, b, c \in \{1, 2, 3\}$ and $a \neq b \neq c$.

A4.1. Using Eqs. (Q4.1.1), (1.1.16) and (1.2.3),

$$\begin{aligned}
\left[\hat{L}_{ab}, \hat{L}_{bc}\right] &= \left[\left(\hat{x}_a\hat{p}_b - \hat{x}_b\hat{p}_a\right), \left(\hat{x}_b\hat{p}_c - \hat{x}_c\hat{p}_b\right)\right] \\
&= \left[\hat{x}_a\hat{p}_b, \hat{x}_b\hat{p}_c\right] - \left[\hat{x}_a\hat{p}_b, \hat{x}_c\hat{p}_b\right] - \left[\hat{x}_b\hat{p}_a, \hat{x}_b\hat{p}_c\right] + \left[\hat{x}_b\hat{p}_a, \hat{x}_c\hat{p}_b\right] \\
&= \hat{x}_a\hat{p}_c\left[\hat{p}_b, \hat{x}_b\right] + \hat{x}_c\hat{p}_a\left[\hat{x}_b, \hat{p}_b\right] \\
&= i\hbar\hat{L}_{ca} \qquad \text{(A4.1.1)}
\end{aligned}$$

and, using Eqs. (1.1.16) and (Q4.1.3),

$$\begin{aligned}
\left[\hat{L}_{ab}, \hat{L}_{bc}^2\right] &= \left[\hat{L}_{ab}, \hat{L}_{bc}\right]\hat{L}_{bc} - \hat{L}_{bc}\left[\hat{L}_{bc}, \hat{L}_{ab}\right] \\
&= i\hbar\hat{L}_{ca}\hat{L}_{bc} + i\hbar\hat{L}_{bc}\hat{L}_{ca} \\
&= \hat{L}_{ca}\left[\hat{L}_{ca}, \hat{L}_{ab}\right] + \left[\hat{L}_{ca}, \hat{L}_{ab}\right]\hat{L}_{ca} \\
&= -\left[\hat{L}_{ab}, \hat{L}_{ca}^2\right] \qquad \text{(A4.1.2)}
\end{aligned}$$

therefore

$$\left[\hat{L}_{ab}, \hat{L}^2\right] = \left[\hat{L}_{ab}, \left(\hat{L}_{ab}^2 + \hat{L}_{bc}^2 + \hat{L}_{ca}^2\right)\right] = 0 \tag{A4.1.3}$$

Q4.2. Determine the eigenvalues of $\hat{L}^2$ and $\hat{L}_{12}$ by considering the properties of

$$\hat{L}_\pm = \hat{L}_{23} \pm i\hat{L}_{31} \tag{Q4.2.1}$$

in a manor similar to Section (1.2.4).

A4.2. $\hat{L}^2$ and $\hat{L}_{12}$ are commuting Hermitian operators and so have a complete set of common eigenvectors

$$\begin{aligned}
\hat{L}^2\left|\lambda, \mu\right\rangle &= \lambda\left|\lambda, \mu\right\rangle \qquad \text{(A4.2.1)} \\
\hat{L}_{12}\left|\lambda, \mu\right\rangle &= \mu\left|\lambda, \mu\right\rangle \qquad \text{(A4.2.2)}
\end{aligned}$$

Eqs. (Q4.2.1) and (Q4.1.3) give

$$\left[\hat{L}_{12}, \hat{L}_{\pm}\right] = i\hbar \left(\hat{L}_{31} \mp i\hat{L}_{23}\right) = \pm\hbar\hat{L}_{\pm} \tag{A4.2.3}$$

therefore

$$\hat{L}_{12}\hat{L}_{\pm} = \hat{L}_{\pm} \left(\hat{L}_{12} \pm \hbar\right) \tag{A4.2.4}$$

therefore, using Eq. (A4.2.2),

$$\hat{L}_{\pm} |\lambda, \mu\rangle \propto |\lambda, \mu \pm \hbar\rangle \tag{A4.2.5}$$

similar to Eqs. (1.2.40) and (1.2.41).

Eqs. (Q4.2.1), (Q4.1.3) and (Q4.1.2) give

$$\begin{aligned} \hat{L}_{\pm}^{\dagger}\hat{L}_{\pm} &= \left(\hat{L}_{23} \mp i\hat{L}_{31}\right) \left(\hat{L}_{23} \pm i\hat{L}_{31}\right) \\ &= \hat{L}_{23}^2 + \hat{L}_{31}^2 \mp \hbar\hat{L}_{12} \\ &= \hat{L}^2 - \hat{L}_{12} \left(\hat{L}_{12} \pm \hbar\right) \end{aligned} \tag{A4.2.6}$$

therefore, using Eqs. (A4.2.1) and (A4.2.2),

$$\left|\hat{L}_{\pm} |\lambda, \mu\rangle\right|^2 = \{\lambda - \mu(\mu \pm \hbar)\} \langle\lambda, \mu|\lambda, \mu\rangle \tag{A4.2.7}$$

therefore

$$\mu(\mu \pm \hbar) \leq \lambda \tag{A4.2.8}$$

and

$$\hat{L}_{\pm} |\lambda, \mu\rangle = 0 \Longleftrightarrow \mu(\mu \pm \hbar) = \lambda \tag{A4.2.9}$$

similar to Eqs. (1.2.43) and (1.2.44).

Eq. (A4.2.8) implies

$$-\frac{\sqrt{4\lambda + \hbar^2} - \hbar}{2} \leq \mu \leq \frac{\sqrt{4\lambda + \hbar^2} - \hbar}{2} \tag{A4.2.10}$$

therefore, using Eqs. (A4.2.5) and (A4.2.9),

$$\sqrt{4\lambda + \hbar^2} \in \{\hbar, 2\hbar, 3\hbar, \ldots\} \tag{A4.2.11}$$

and

$$\mu = -\frac{\sqrt{4\lambda + \hbar^2} - \hbar}{2}, \ldots, \frac{\sqrt{4\lambda + \hbar^2} - 3\hbar}{2}, \frac{\sqrt{4\lambda + \hbar^2} - \hbar}{2} \tag{A4.2.12}$$

Parametrizing the eigenvalues as

$$\sqrt{4\lambda + \hbar^2} = (2l + 1)\hbar \tag{A4.2.13}$$

i.e.

$$\lambda = l\,(l+1)\,\hbar^2 \tag{A4.2.14}$$

and

$$\mu = m\hbar \tag{A4.2.15}$$

Eqs. (A4.2.11) and (A4.2.12) become

$$l \in \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\right\} \tag{A4.2.16}$$

and

$$m = -l, \ldots, l-1, l \tag{A4.2.17}$$

similar to Eq. (1.2.47).