Chapter 1

Quantum mechanics

1.1 Hilbert spaces

1.1.1 Kets, bras, conjugation and contraction

A **complex number** $z \in \mathbb{C}$ can be written as

$$z = x + iy \tag{1.1.1}$$

where $x, y \in \mathbb{R}$ and $i^2 = -1$. The **complex conjugate** of a complex number z is

$$z^* = x - iy \tag{1.1.2}$$

and its magnitude squared is

$$|z|^2 = z^* z = x^2 + y^2 \tag{1.1.3}$$

A **vector space** is a set whose elements, called vectors, can be added, and multiplied by a scalar, in the usual way. We will consider complex vector spaces, in which case the scalars are complex numbers, and use quantum mechanics notation which denotes a vector ϕ by $|\phi\rangle$ and calls it a **ket**. Then we have the basic operations of addition

$$|\phi\rangle + |\psi\rangle = |\chi\rangle \tag{1.1.4}$$

and multiplication by a scalar

$$\alpha\, |\phi\rangle = |\xi\rangle \tag{1.1.5}$$

It is natural to extend the concept of complex conjugation to vectors, in which case it is called **Hermitian conjugation** and denoted [1] by a superscript †. The Hermitian conjugate of a ket $|\phi\rangle$ is a **bra** $\langle\phi|$, and vice versa

$$|\phi\rangle^\dagger = \langle\phi| \quad , \quad \langle\phi|^\dagger = |\phi\rangle \tag{1.1.6}$$

[1] Mathematicians simply use a superscript $*$.

The bras form a dual vector space, with Hermitian conjugation providing an antilinear bijection between the kets and the bras

$$(\alpha |\phi\rangle + \beta |\psi\rangle)^\dagger = \alpha^* \langle\phi| + \beta^* \langle\psi| \tag{1.1.7}$$

Bras and kets can be **contracted** together to give a **bra**c**ket** which is a scalar

$$\langle\phi|\psi\rangle \in \mathbb{C} \tag{1.1.8}$$

with

$$\langle\psi|\phi\rangle = (\langle\phi|\psi\rangle)^\dagger = (\langle\phi|\psi\rangle)^* \tag{1.1.9}$$

Combining Hermitian conjugation with contraction gives the **magnitude** squared of a ket

$$|\phi|^2 = |\phi\rangle^\dagger |\phi\rangle = \langle\phi|\phi\rangle > 0 \qquad \text{for } |\phi\rangle \neq 0 \tag{1.1.10}$$

More generally, the **inner product** of two kets $|\phi\rangle$ and $|\psi\rangle$ is defined by

$$|\phi\rangle^\dagger |\psi\rangle = \langle\phi|\psi\rangle \tag{1.1.11}$$

They are said to be **orthogonal** if $\langle\phi|\psi\rangle = 0$.

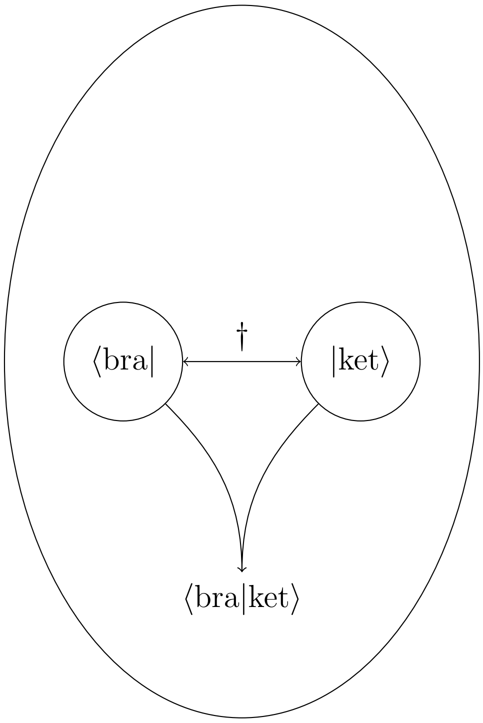

Figure 1.1.1: Hilbert's face.

1.1.2 Linear operators

Linear operators are linear mappings from the Hilbert space to itself.

$$A|\phi\rangle = |\psi\rangle \tag{1.1.12}$$

The **Hermitian conjugate** $A^\dagger$ of an operator A is defined by

$$\langle\psi| A^\dagger |\phi\rangle = (\langle\phi| A |\psi\rangle)^\dagger = (\langle\phi| A |\psi\rangle)^* \tag{1.1.13}$$

consistent with Eq. (1.1.9), and so

$$A^{\dagger\dagger} = A \tag{1.1.14}$$

and

$$(AB)^\dagger = B^\dagger A^\dagger \tag{1.1.15}$$

The **commutator** of two operators A and B is defined by

$$[A, B] = AB - BA \tag{1.1.16}$$

They are said to commute if $[A, B] = 0$.

A **Hermitian operator** H has the property

$$H^\dagger = H \tag{1.1.17}$$

Hermitian operators are an important special case of the more general class of **normal operators**, which have the property

$$\left[N, N^\dagger\right] = 0 \tag{1.1.18}$$

1.1.3 Eigenspaces

An **eigenvector** $|\phi_\alpha\rangle$ of an operator A satisfies

$$A|\phi_\alpha\rangle = \alpha|\phi_\alpha\rangle \tag{1.1.19}$$

where the **eigenvalue** α is a scalar. Any linear combination of eigenvectors with eigenvalue α is also an eigenvector with eigenvalue α, so eigenvectors with the same eigenvalue form a subspace of the Hilbert space called an **eigenspace**

$$\mathcal{A}_\alpha = \{|\phi\rangle : A|\phi\rangle = \alpha|\phi\rangle\} \tag{1.1.20}$$

If A is a normal operator then its eigenspaces $\mathcal{A}_\alpha$ are **orthogonal** and **complete**, i.e. vectors in different eigenspaces are orthogonal

$$\langle\phi_\alpha|\phi_\beta\rangle = 0 \qquad (|\phi_\alpha\rangle \in \mathcal{A}_\alpha \neq \mathcal{A}_\beta \ni |\phi_\beta\rangle) \tag{1.1.21}$$

and any vector in the Hilbert space can be expressed as a sum of vectors from the eigenspaces

$$|\phi\rangle = \sum_\alpha |\phi_\alpha\rangle \qquad (|\phi_\alpha\rangle \in \mathcal{A}_\alpha) \tag{1.1.22}$$

See Figure 1.1.2.

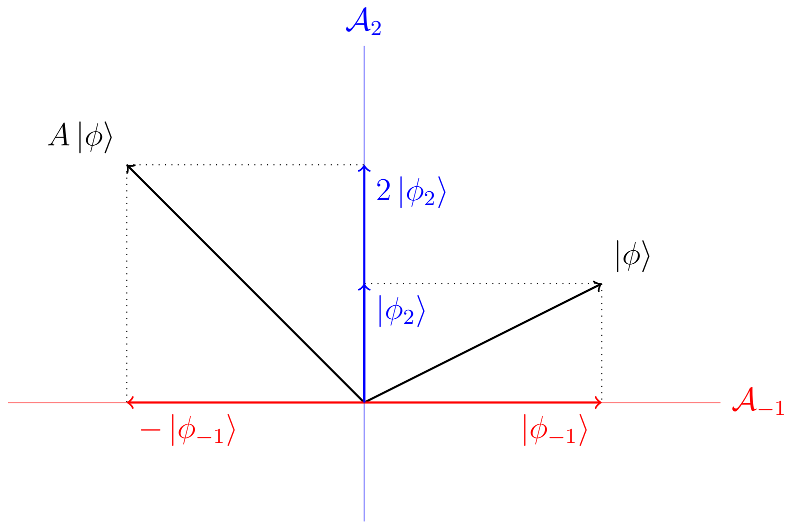

Figure 1.1.2: $A|\phi\rangle = A(|\phi_{-1}\rangle + |\phi_2\rangle) = -|\phi_{-1}\rangle + 2|\phi_2\rangle$.

1.1.4 Quantum interpretation

A general linear operator acting on a general vector

$$A|\phi\rangle = |\psi\rangle \tag{1.1.23}$$

can only be interpreted as a mapping of one state into another. However, a linear operator acting on an **eigenvector**

$$A|\phi_\alpha\rangle = \alpha|\phi_\alpha\rangle \tag{1.1.24}$$

has a clear interpretation: $|\phi_\alpha\rangle$ is the state of the physical system, A is a quantity, and α is the value of that quantity in that state. Note that the magnitude of $|\phi_\alpha\rangle$ has no effect on the value of A.

Since the eigenspaces of a **normal operator** are complete, we can decompose a general vector in terms of the eigenvectors of the normal operator

$$|\phi\rangle = \sum_\alpha |\phi_\alpha\rangle \tag{1.1.25}$$

as in Eq. (1.1.22). Therefore the action of a normal operator on a general vector can be decomposed in terms of its action on its eigenspaces

$$A|\phi\rangle = \sum_\alpha \alpha|\phi_\alpha\rangle \tag{1.1.26}$$

Thus, in a general state, a normal quantity has a set of values, and the state can be regarded as a **superposition** of eigenstates with those values.

Furthermore, since the eigenspaces of a normal operator are orthogonal, see Eq. (1.1.21),

$$\frac{\langle\phi|A|\phi\rangle}{\langle\phi|\phi\rangle} = \sum_\alpha \alpha P_\alpha \tag{1.1.27}$$

Hilbert space	$\mathcal{H}$	physical system
vector	$\|\phi\rangle$	state
linear operator	A	quantity
operator relation	$f(A,B)=0$	law
eigenvalue	α	value of a quantity
eigenvector	$\|\phi_\alpha\rangle$	state with a definite value of a quantity
eigenspace	$\mathcal{A}_\alpha$	set of states with the same definite value of a quantity
normal operator	N	quantity that always has values
Hermitian operator	H	quantity that always has real values
dimension	D	maximum number of values a quantity can have
addition	$\|\phi\rangle + \|\psi\rangle$	superposition of states
contraction	$\langle\phi\|\psi\rangle$	overlap between states
orthogonal	$\langle\phi\|\psi\rangle = 0$	distinct states
conjugate	$\dagger$	
bra	$\langle\phi\|$	
magnitude	$\langle\phi\|\phi\rangle$	
multiplication	$\alpha\,\|\phi\rangle$	

Table 1.1.1: Hilbert space - physics dictionary

where

$$P_\alpha = \frac{\langle\phi_\alpha|\phi_\alpha\rangle}{\langle\phi|\phi\rangle} \tag{1.1.28}$$

can be interpreted as a relative weighting of the different eigenvalues. Thus, in state $|\phi\rangle$, A has all the values α with weightings P_α, and $\langle\phi|\,A\,|\phi\rangle \,/\, \langle\phi|\phi\rangle$ is the weighted average or **expectation value** of A in state $|\phi\rangle$.