1.3 Shrödinger picture

1.3.1 Shrödinger's equation

In the Shrödinger picture we start with an extended Hilbert space, containing both physical and unphysical states, corresponding to the embeddings of a particle in space-time. In this picture, the position and time of the particle are independent quantities, and we get the following commutation relations

$$\left[\hat{x},\hat{t}\right] = 0 \tag{1.3.1}$$

$$\left[\hat{p},\hat{E}\right] = 0 \tag{1.3.2}$$

and

$$\begin{array}{rclcrcl} \left[\hat{x},\hat{p}\right] & = & i\hbar & , & \left[\hat{x},\hat{E}\right] & = & 0 \\ \left[\hat{t},\hat{p}\right] & = & 0 & , & \left[\hat{t},\hat{E}\right] & = & -i\hbar \end{array} \tag{1.3.3}$$

We then apply the following constraint, called **Schrödinger's equation**, to the Hilbert space

$$\hat{E}\left|\psi\right\rangle = \hat{H}\left(\hat{x},\hat{p},\hat{t}\right)\left|\psi\right\rangle \tag{1.3.4}$$

to obtain the physical states $|\psi\rangle$.

Decomposing $|\psi\rangle$ into energy eigenstates [1]

$$\left|\psi\right\rangle = \sum_E \left|\psi_E\right\rangle \tag{1.3.5}$$

with

$$\hat{E}\left|\psi_E\right\rangle = E\left|\psi_E\right\rangle \tag{1.3.6}$$

Schrödinger's equation reduces to

$$\hat{H}\left|\psi_E\right\rangle = E\left|\psi_E\right\rangle \tag{1.3.7}$$

1.3.2 Wave function

The **wave function** of a state is the components of the state vector with respect to a basis. For example, it is often convenient to choose a basis of eigenvectors of $\hat{x}$ and $\hat{t}$ [2]

$$\hat{x}\left|x,t\right\rangle = x\left|x,t\right\rangle \tag{1.3.8}$$

$$\hat{t}\left|x,t\right\rangle = t\left|x,t\right\rangle \tag{1.3.9}$$

Then the wave function of a state $|\psi\rangle$ is

$$\psi(x,t) \equiv \langle x,t|\psi\rangle \tag{1.3.10}$$

[1] Or $\int dE\,|\psi(E)\rangle$ for a continuous range of E.

[2] Note that the eigenvectors $|x,t\rangle$, corresponding to the particle existing only at the position x and time t, are not physical states, but any physical history of the particle can be constructed from them.

Note that the wavefunction is related to the weighting of the eigenvalues in Eq. (1.1.28) by

$$|\psi(x,t)|^2 \propto P(x,t) \tag{1.3.11}$$

We can also reexpress linear operators in components. For example, Eqs. (1.3.8) and (1.3.9) give

$$\langle x,t|\,\hat{x}\,|\psi\rangle = x\,\psi(x,t) \tag{1.3.12}$$

$$\langle x,t|\,\hat{t}\,|\psi\rangle = t\,\psi(x,t) \tag{1.3.13}$$

and Eqs. (1.3.3) are satisfied by

$$\langle x,t|\,\hat{p}\,|\psi\rangle = -i\hbar\frac{\partial}{\partial x}\,\psi(x,t) \tag{1.3.14}$$

$$\langle x,t|\,\hat{E}\,|\psi\rangle = i\hbar\frac{\partial}{\partial t}\,\psi(x,t) \tag{1.3.15}$$

Using Eqs. (1.3.12) to (1.3.15) and taking

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) \tag{1.3.16}$$

Eq. (1.3.4) becomes the **Schrödinger wave equation** for a non-relativistic particle [3]

$$i\hbar\frac{\partial}{\partial t}\,\psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi(x,t) \tag{1.3.17}$$

The energy eigenstates have wave functions $\psi_E(x,t) \equiv \langle x,t|\psi_E\rangle$ with time dependence determined by Eqs. (1.3.6) and (1.3.15)

$$0 = \langle x,t|\left(\hat{E} - E\right)|\psi_E\rangle = \left(i\hbar\frac{\partial}{\partial t} - E\right)\psi_E(x,t) \tag{1.3.18}$$

therefore

$$\psi_E(x,t) \propto \exp\left(\frac{-iEt}{\hbar}\right) \tag{1.3.19}$$

and spatial dependence determined by Eqs. (1.3.7), (1.3.16) and (1.3.14)

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi_E(x,t) = E\,\psi_E(x,t) \tag{1.3.20}$$

[3] Note that, as in Section 1.3.1, we start with a general function of x and t and use the Schrödinger wave equation to restrict to physical wave functions.

1.3.3 Free non-relativistic particle

Continuing from Section 1.2.3, the spatial dependence of the momentum eigenfunctions

$$\psi_p(x,t) \equiv \langle x,t|\psi_p\rangle \tag{1.3.21}$$

can be determined using Eqs. (1.2.13) and (1.3.14)

$$0 \;=\; \langle x,t|\left(\hat{p}-p\right)|\psi_p\rangle \;=\; \left(-i\hbar\frac{\partial}{\partial x}-p\right)\psi_p(x,t) \tag{1.3.22}$$

therefore

$$\psi_p(x,t) \propto \exp\left(\frac{ipx}{\hbar}\right) \tag{1.3.23}$$

and their time dependence is given by Eq. (1.3.19) with

$$E = \frac{p^2}{2m} \tag{1.3.24}$$

Thus the wave function of a free non-relativistic particle is [4]

$$\psi(x,t) = \sum_p c_p \exp\left(\frac{ipx}{\hbar}\right)\exp\left(\frac{-iEt}{\hbar}\right) \tag{1.3.25}$$

for some constants c_p. This corresponds to a superposition of plane waves with wavenumber

$$k = \frac{p}{\hbar} \tag{1.3.26}$$

and frequency

$$\omega = \frac{E}{\hbar} \tag{1.3.27}$$

1.3.4 Simple harmonic oscillator

Continuing from Section 1.2.4, the spatial dependence of the eigenfunctions

$$\psi_n(x,t) \equiv \langle x,t|n\rangle \tag{1.3.28}$$

can be determined using Eqs. (1.2.44), (1.2.27), (1.2.24) and (1.3.14)

$$0 \;=\; \langle x,t|\,\hat{a}\,|0\rangle \;=\; \langle x,t|\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}+\frac{i\hat{p}}{m\omega}\right)e^{i\omega t}|0\rangle \tag{1.3.29}$$

$$\;=\; \sqrt{\frac{m\omega}{2\hbar}}\,e^{i\omega t}\left(x+\frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\psi_0(x,t) \tag{1.3.30}$$

therefore

$$\psi_0(x,t) \propto \exp\left(-\frac{m\omega}{2\hbar}x^2\right) \tag{1.3.31}$$

[4] Or $\int dp\, c(p)\exp\left(ipx/\hbar\right)\exp\left(-iEt/\hbar\right)$ for a continuous range of p.

and using Eq. (1.2.41)

$$\begin{aligned}
\psi_n(x,t) &= \langle x,t|n\rangle &(1.3.32)\\
&\propto \langle x,t|\left(\hat{a}^\dagger\right)^n|0\rangle &(1.3.33)\\
&= \langle x,t|\left\{\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}-\frac{i\hat{p}}{m\omega}\right)e^{-i\omega t}\right\}^n|0\rangle &(1.3.34)\\
&= \left(\frac{m\omega}{2\hbar}\right)^{\frac{n}{2}}e^{-in\omega t}\left(x-\frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)^n\psi_0(x,t) &(1.3.35)\\
&\propto \left(x-\frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)^n\exp\left(-\frac{m\omega}{2\hbar}x^2\right) &(1.3.36)
\end{aligned}$$

The time dependence is given by Eq. (1.3.19) and (1.2.33),

$$\psi_n(x,t)\propto\exp\left[-i\left(n+\frac{1}{2}\right)\omega t\right] \tag{1.3.37}$$

Thus the wavefunction of a simple harmonic iscillator is

$$\psi(x,t)=\sum_{n=0}^{\infty}c_n\left[\left(x-\frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)^n\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\right]\exp\left[-i\left(n+\frac{1}{2}\right)\omega t\right] \tag{1.3.38}$$

for some constants c_n.