2.2 Maxwell's equations

2.2.1 Relativistic particle and field

Electrodynamics is a relativistic theory and so is most simply described in spacetime. A worldline C in a spacetime has action

$$-S[C] = \int_C \left(m\underline{\sigma} + q\underline{A}\right) \tag{2.2.1}$$

where the worldline volume form $\underline{\sigma}$ measures the oriented volume of, i.e. the length along, the worldline, the **electromagnetic potential** $\underline{A}$ is a one-form field in the spacetime and the constants m and q are the mass and charge. See Figure 2.2.1.

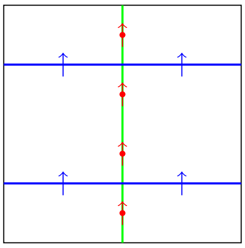


Figure 2.2.1: A worldline in a spacetime with worldline volume form and spacetime one-form field. Two dimensions external to the worldline have been suppressed.

In Lagrangian form

$$\begin{aligned} -S &= \int_C \left(m\sigma_{\mathbf{a}} + qA_{\mathbf{a}}\right) dx^{\mathbf{a}} && (2.2.2)\\ &= \int_C \left(m\sigma_{\mathbf{a}} + qA_{\mathbf{a}}\right) \dot{x}^{\mathbf{a}}\, dt && (2.2.3)\\ &= \int_C \left(m\sqrt{g_{\mathbf{ab}}\dot{x}^{\mathbf{a}}\dot{x}^{\mathbf{b}}} + qA_{\mathbf{a}}\dot{x}^{\mathbf{a}}\right) dt && (2.2.4)\end{aligned}$$

The momentum is

$$p_{\mathbf{a}} = -\frac{\partial L}{\partial \dot{x}^{\mathbf{a}}} = \frac{m g_{\mathbf{ab}}\dot{x}^{\mathbf{b}}}{\sqrt{g_{\mathbf{cd}}\dot{x}^{\mathbf{c}}\dot{x}^{\mathbf{d}}}} + qA_{\mathbf{a}} = m g_{\mathbf{ab}}\frac{dx^{\mathbf{b}}}{d\tau} + qA_{\mathbf{a}} \tag{2.2.5}$$

where τ is the proper time

$$d\tau^2 = g_{\mathbf{ab}}\, dx^{\mathbf{a}}\, dx^{\mathbf{b}} \tag{2.2.6}$$

The Euler-Lagrange equation is

$$\frac{dp_{\mathbf{a}}}{d\tau} = q\left(\nabla_{\mathbf{a}}A_{\mathbf{b}}\right)\frac{dx^{\mathbf{b}}}{d\tau} \tag{2.2.7}$$

or

$$m g_{\mathbf{ab}}\frac{d^2x^{\mathbf{b}}}{d\tau^2} = q\left(\nabla_{\mathbf{a}}A_{\mathbf{b}} - \nabla_{\mathbf{b}}A_{\mathbf{a}}\right)\frac{dx^{\mathbf{b}}}{d\tau} \tag{2.2.8}$$

Thus the **Lorentz force** acting on a charged particle is

$$\underline{f} = q\underline{\underline{F}} \cdot \vec{u} \tag{2.2.9}$$

where the **electromagnetic field**

$$\underline{\underline{F}} = \underline{\nabla} \wedge \underline{A} \tag{2.2.10}$$

and the velocity of the particle

$$\vec{u} = \frac{d\vec{x}}{d\tau} \tag{2.2.11}$$

See Figure 2.2.2.

Figure 2.2.2: The diagram below represents the spacetime Lorentz force $\underline{f}$ acting on a particle with charge velocity $q\vec{u}$ due to an electromagnetic field $\underline{\underline{F}}$, see Eq. (2.2.9). Two spacetime dimensions internal to $\underline{f}$ and $\underline{\underline{F}}$ and external to $q\vec{u}$ have been suppressed.

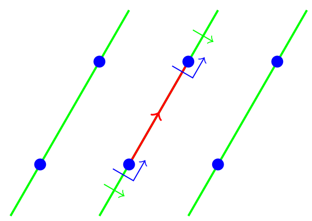


Figure 2.2.3: The diagram above also represents the spatial Lorentz force $\underline{F}$ acting on a particle with charge velocity $q\vec{v}$ due to an magnetic field $\underline{\underline{B}}$, see Eq. (2.2.17). In this case one space dimension internal to $\underline{F}$ and $\underline{\underline{B}}$ and external to $q\vec{v}$ has been suppressed.

Our usual space and time description of physics necessarily complicates the above equations. Defining a choice of spacial hypersurfaces in terms of a time coordinate t

$$\underline{t} = \underline{\nabla} \wedge t \tag{2.2.12}$$

decomposing $\underline{f}$ into the **electromagnetic power** P and spatial **Lorentz force** $\underline{F}$

$$\underline{f} = \frac{dt}{d\tau}\left(P\underline{t} - \underline{F}\right) \tag{2.2.13}$$

$\underline{\underline{F}}$ into the **electric field** $\underline{E}$ and the **magnetic flux density** $\underline{\underline{B}}$

$$\underline{\underline{F}} = \underline{t} \wedge \underline{E} - \underline{\underline{B}} \tag{2.2.14}$$

and $\vec{u}$ into a time vector $\vec{t}$, satisfying $\vec{t} \cdot \underline{t} = 1$, and the spacial velocity $\vec{v}$

$$\vec{u} = \frac{dt}{d\tau}\left(\vec{t} + \vec{v}\right) \tag{2.2.15}$$

Eq. (2.2.9) gives

$$P = q\underline{E} \cdot \vec{v} \tag{2.2.16}$$

$$\underline{F} = q\left(\underline{E} + \underline{\underline{B}} \cdot \vec{v}\right) \tag{2.2.17}$$

see Figure 2.2.3.

2.2.2 Electromagnetic field

The electromagnetic field $\underline{\underline{F}}$ is the exterior derivative of the electromagnetic potential $\underline{A}$, see Eq. (2.2.10) and Figure 2.2.4, and hence satisfies Maxwell's first equation

$$\underline{\nabla} \wedge \underline{\underline{F}} = 0 \qquad (2.2.18)$$

i.e. the electromagnetic field surfaces have no boundaries. Note that $\underline{\underline{F}}$, as well as the

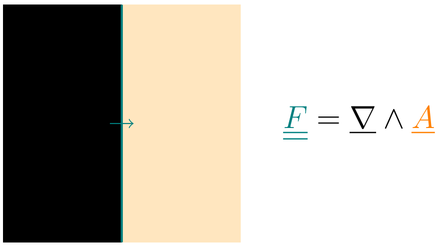


Figure 2.2.4: Electromagnetic field induced by electromagnetic potential. One external and one internal spacetime dimension have been suppressed.

action in Eq. (2.2.1), are invariant under the **gauge transformation**

$$\underline{A} \to \underline{A} + \underline{\nabla} \wedge \lambda \qquad (2.2.19)$$

Decomposing the spacetime derivative into the time and space derivatives

$$\underline{\nabla_4} \wedge = \underline{t} \wedge \frac{\partial}{\partial t} + \underline{\nabla_3} \wedge \qquad (2.2.20)$$

the electromagnetic potential $\underline{A_4}$ into the **electric potential** ϕ and the **magnetic** "vector" **potential** $\underline{A_3}$

$$\underline{A_4} = \phi \underline{t} - \underline{A_3} \qquad (2.2.21)$$

and $\underline{\underline{F}}$ into $\underline{E}$ and $\underline{\underline{B}}$ as in Eq. (2.2.14), see Figure 2.2.5, Eq. (2.2.10) decomposes to

$$\underline{E} = -\underline{\nabla} \wedge \phi - \frac{\partial}{\partial t} \underline{A} \qquad (2.2.22)$$

$$\underline{\underline{B}} = \underline{\nabla} \wedge \underline{A} \qquad (2.2.23)$$

which are inariant under the decomposition of Eq. (2.2.19)

$$\phi \to \phi + \dot{\lambda} \qquad (2.2.24)$$

$$\underline{A} \to \underline{A} - \underline{\nabla} \wedge \lambda \qquad (2.2.25)$$

As in Eqs. (2.2.10) and (2.2.18), Eqs. (2.2.22) and (2.2.23) imply the first half of **Maxwell's equations**

$$\underline{\nabla} \wedge \underline{E} + \frac{\partial}{\partial t} \underline{\underline{B}} = 0 \qquad (2.2.26)$$

$$\underline{\nabla} \wedge \underline{\underline{B}} = 0 \qquad (2.2.27)$$

see Figure 2.2.6.

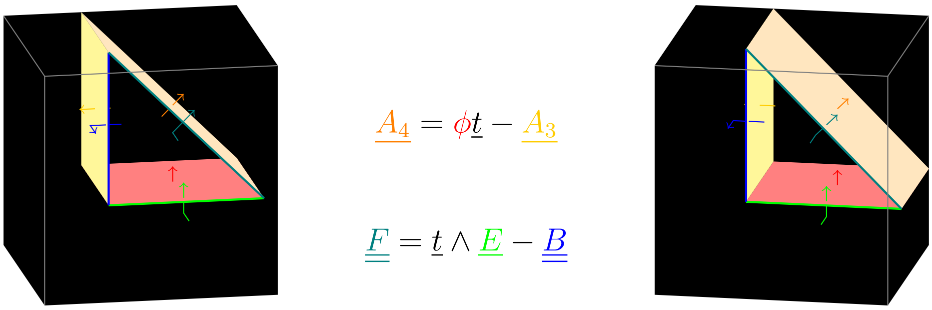
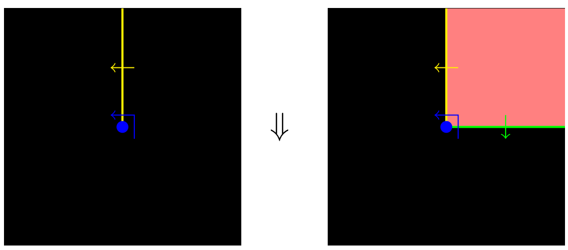


Figure 2.2.5: Space and time decomposition of the electromagnetic field and potential. One internal space dimension has been suppressed.

$\underline{\underline{B}} = \underline{\nabla} \wedge \underline{A}$ & $\underline{E} = -\underline{\nabla}\phi - \frac{\partial}{\partial t}\underline{A}$



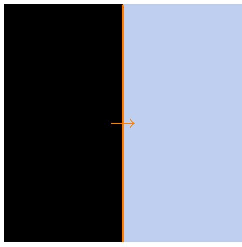
$\Downarrow$

$\underline{\nabla} \wedge \underline{\underline{B}} = 0$ & $\underline{\nabla} \wedge \underline{E} + \frac{\partial}{\partial t}\underline{\underline{B}} = 0$

Figure 2.2.6: Electric field and magnetic flux induced by electric and magnetic potentials. One internal space dimension has been suppressed.

2.2.3 Electromagnetic flux

A charge's worldline corresponds to a spacetime **current density** $\underline{\underline{\underline{J}}}$ flux line. This acts as a source for the **electromagnetic flux density** $\underline{\underline{G}}$, whose two dimensional flux surfaces emerge from the current density flux lines, see Figure 2.2.7. This is expressed



$$\underline{\nabla} \wedge \underline{\underline{G}} = \underline{\underline{\underline{J}}}$$

Figure 2.2.7: Electromagnetic flux induced by spacetime current. Two spacetime dimensions external to the flux and current have been suppressed.

by Maxwell's second equation

$$\underline{\nabla} \wedge \underline{\underline{G}} = \underline{\underline{\underline{J}}} \tag{2.2.28}$$

Since the boundary of a boundary is zero, Eq. (2.2.28) implies

$$\underline{\nabla} \wedge \underline{\underline{\underline{J}}} = 0 \tag{2.2.29}$$

i.e. that the charge worldlines have no ends, corresponding to charge conservation.

Decomposing $\underline{\underline{G}}$ into the **magnetic field** $\underline{H}$ and the **electric flux density** $\underline{\underline{D}}$

$$\underline{\underline{G}} = -\underline{t} \wedge \underline{H} - \underline{\underline{D}} \tag{2.2.30}$$

and $\underline{\underline{\underline{J}}}$ into the spatial **current density** $\underline{\underline{j}}$ and the **charge density** $\underline{\underline{\underline{\rho}}}$

$$\underline{\underline{\underline{J}}} = \underline{t} \wedge \underline{\underline{j}} - \underline{\underline{\underline{\rho}}} \tag{2.2.31}$$

see Figure 2.2.8, Eq. (2.2.28) decomposes to the second half of **Maxwell's equations**

$$\underline{\nabla} \wedge \underline{\underline{D}} = \underline{\underline{\underline{\rho}}} \tag{2.2.32}$$

$$\underline{\nabla} \wedge \underline{H} - \frac{\partial}{\partial t}\underline{\underline{D}} = \underline{\underline{j}} \tag{2.2.33}$$

see Figure 2.2.9. As in Eq. (2.2.29), these equations imply charge conservation

$$\frac{\partial}{\partial t}\underline{\underline{\underline{\rho}}} + \underline{\nabla} \wedge \underline{\underline{j}} = 0 \tag{2.2.34}$$

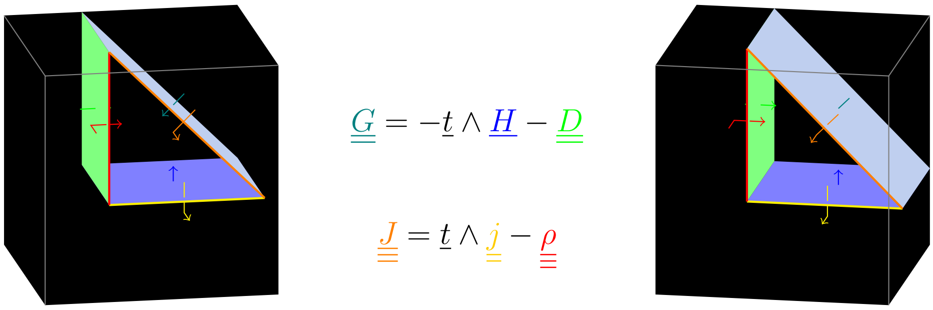


Figure 2.2.8: Space and time decomposition of the electromagnetic flux and spacetime current. One external space dimension has been suppressed.

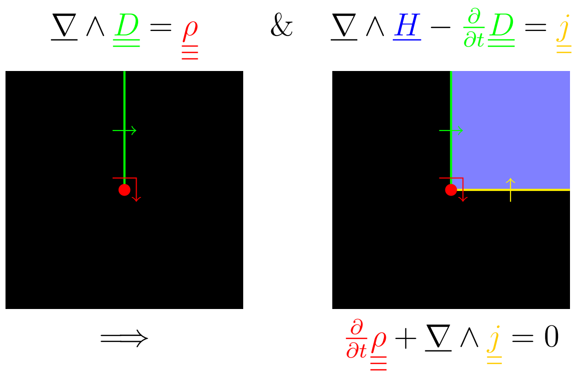


Figure 2.2.9: Electric flux and magnetic field induced by charge and current. One external space dimension has been suppressed.

2.2.4 Permittivity and permeability

The relationship between the electromagnetic field and flux density can be complicated in a general medium but is relatively simple in vacuum in natural units

$$\underline{\underline{G}} = *\underline{\underline{F}} \tag{2.2.35}$$

where $*$ is the Hodge dual, which involves both the volume form and the metric, and has the meaning that $\underline{\underline{F}}$ and $\underline{\underline{G}}$ are orthogonal and have the same magnitude, see Figure 2.2.10.

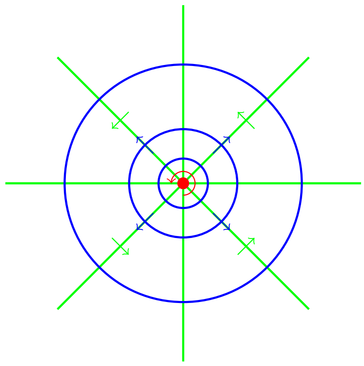


Figure 2.2.10: The electromagnetic flux density $\underline{\underline{G}}$ and field $\underline{\underline{F}}$ generated by a spacetime current $\underline{\underline{\underline{J}}}$ in vacuum. Two spacetime dimensions, one internal and one external to $\underline{\underline{F}}$ and external and internal to $\underline{\underline{G}}$, have been suppressed. The relative orientation of $\underline{\underline{F}}$ and $\underline{\underline{G}}$ is determined by the spacetime orientation.

Again, the relationship between the electric and magnetic fields and flux densities can be complicated in a general medium but is relatively simple in vacuum

$$\underline{\underline{D}} = \varepsilon_0 *\underline{E} \tag{2.2.36}$$

$$\underline{H} = \mu_0^{-1} *\underline{\underline{B}} \tag{2.2.37}$$

see Figures 2.2.11 and 2.2.12, where ε_0 is the vacuum **permittivity** and μ_0 is the vacuum **permeability**, which satisfy

$$\varepsilon_0 \mu_0 c^2 = 1 \tag{2.2.38}$$

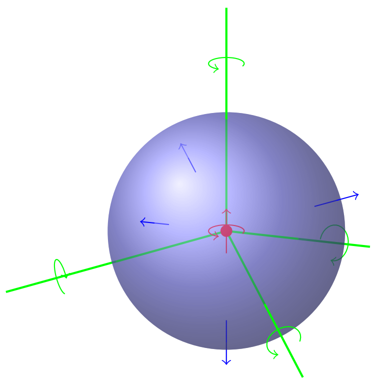


Figure 2.2.11: The electric flux density $\underline{\underline{D}}$ and field $\underline{E}$ generated by a charge $\underline{\underline{\underline{\rho}}}$ in vacuum. The relative orientation of $\underline{\underline{D}}$ and $\underline{E}$ is determined by the spatial orientation, which is usually chosen to be right-handed.

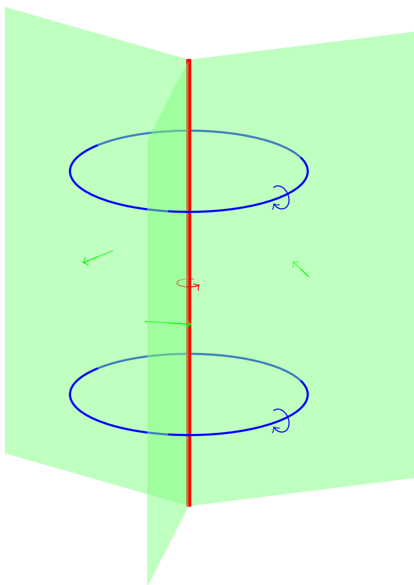


Figure 2.2.12: The magnetic field $\underline{H}$ and flux density $\underline{\underline{B}}$ generated by a current $\underline{\underline{j}}$ in vacuum. The relative orientation of $\underline{H}$ and $\underline{\underline{B}}$ is determined by the spatial orientation, which is usually chosen to be right-handed.

In a medium, it is convenient to reexpress the local bound currents in terms of the polarization and magnetization of the medium

$$\underline{\underline{\underline{J}}} = \underline{\underline{\underline{J_{\mathrm{f}}}}} + \underline{\underline{\underline{J_{\mathrm{b}}}}} \tag{2.2.39}$$

with the bound current

$$\underline{\underline{\underline{J_{\mathrm{b}}}}} = \underline{\nabla} \wedge \underline{\underline{N}} \tag{2.2.40}$$

where $\underline{\underline{N}}$ is the polarization-magnetization tensor. By definition, the free and bound currents do not mix and so are separately conserved. Maxwell's second equation is then expressed in terms of the free current

$$\underline{\nabla} \wedge \underline{\underline{G}} = \underline{\underline{\underline{J_{\mathrm{f}}}}} \tag{2.2.41}$$

with

$$\underline{\underline{G}} = *\underline{\underline{F}} - \underline{\underline{N}} \tag{2.2.42}$$

in natural units.

Decomposing into space and time components

$$\underline{\underline{N}} = -\underline{t} \wedge \underline{M} + \underline{\underline{P}} \tag{2.2.43}$$

where $\underline{\underline{P}}$ is the electric dipole moment density or **polarization** and $\underline{M}$ is the magnetic dipole moment density or **magnetization** of the medium. The second half of Maxwell's equations becomes

$$\underline{\nabla} \wedge \underline{\underline{D}} = \underline{\underline{\underline{\rho_{\mathrm{f}}}}} \tag{2.2.44}$$

$$\underline{\nabla} \wedge \underline{H} - \frac{\partial}{\partial t}\underline{\underline{D}} = \underline{\underline{j_{\mathrm{f}}}} \tag{2.2.45}$$

with

$$\underline{\underline{D}} = \varepsilon_0 *\underline{E} + \underline{\underline{P}} \tag{2.2.46}$$

$$\underline{H} = \mu_0^{-1} *\underline{\underline{B}} - \underline{M} \tag{2.2.47}$$