Homework 2

Answers should be submitted, as both a tex file and a pdf file, to both me and the teaching assistants. You may use this file as a template.

Q2.1. Let $f(z, z^*)$ be a complex function and $\boldsymbol{f}$ be the vector field on the complex plane with Cartesian components $(\operatorname{Re} f, \operatorname{Im} f)$. Show that the conditions

$$f = f(z^*) \tag{Q2.1.1}$$

and

$$\boldsymbol{\nabla} \wedge \boldsymbol{f} = 0 = \boldsymbol{\nabla} \cdot \boldsymbol{f} \tag{Q2.1.2}$$

are equivalent.

Q2.2. Let $f(z, z^*)$ be a complex function. Show that

$$(\nabla \operatorname{Re} f) \cdot (\nabla \operatorname{Im} f) = 0 = (\nabla \operatorname{Re} f)^2 - (\nabla \operatorname{Im} f)^2 \tag{Q2.2.1}$$

if and only if f is a holomorphic or antiholomorphic function, with the holomorphicity or antiholomorphicity determined by the sign of

$$(\nabla \operatorname{Re} f) \wedge (\nabla \operatorname{Im} f) \tag{Q2.2.2}$$

What are the meanings of Eqs. (Q2.2.1) and (Q2.2.2)?

Q2.3. Consider the coordinates $w = u + iv$ defined relative to the Cartesian coordinates $z = x + iy$ by the holomorphic function

$$z = f(w) \tag{Q2.3.1}$$

(a) What properties do the coordinates (u, v) have?

(b) Express the Laplacian in terms of w and hence u and v.

(c) Use PGF to draw the coordinates (u, v) on the (x, y) plane in the case

$$z = \cosh w \tag{Q2.3.2}$$