Homework 2

Q2.1. Let $f(z,z^*)$ be a complex function and $\boldsymbol{f}$ be the vector field on the complex plane with Cartesian components $(\operatorname{Re} f, \operatorname{Im} f)$. Show that the conditions

$$f = f(z^*) \tag{Q2.1.1}$$

and

$$\boldsymbol{\nabla} \wedge \boldsymbol{f} = 0 = \boldsymbol{\nabla} \cdot \boldsymbol{f} \tag{Q2.1.2}$$

are equivalent.

A2.1.

$$\begin{aligned}
\frac{\partial f}{\partial z} &= \frac{\partial x}{\partial z}\frac{\partial \operatorname{Re} f}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial \operatorname{Re} f}{\partial y} + i\frac{\partial x}{\partial z}\frac{\partial \operatorname{Im} f}{\partial x} + i\frac{\partial y}{\partial z}\frac{\partial \operatorname{Im} f}{\partial y} && \text{(A2.1.1)}\\
&= \frac{1}{2}\left(\frac{\partial \operatorname{Re} f}{\partial x} + \frac{\partial \operatorname{Im} f}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial \operatorname{Im} f}{\partial x} - \frac{\partial \operatorname{Re} f}{\partial y}\right) && \text{(A2.1.2)}\\
&= \frac{1}{2}\boldsymbol{\nabla} \cdot \boldsymbol{f} + \frac{i}{2}\boldsymbol{\nabla} \wedge \boldsymbol{f} && \text{(A2.1.3)}
\end{aligned}$$

Q2.2. Let $f(z,z^*)$ be a complex function. Show that

$$(\nabla \operatorname{Re} f) \cdot (\nabla \operatorname{Im} f) = 0 = (\nabla \operatorname{Re} f)^2 - (\nabla \operatorname{Im} f)^2 \tag{Q2.2.1}$$

if and only if f is a holomorphic or antiholomorphic function, with the holomorphicity or antiholomorphicity determined by the sign of

$$(\nabla \operatorname{Re} f) \wedge (\nabla \operatorname{Im} f) \tag{Q2.2.2}$$

What are the meanings of Eqs. (Q2.2.1) and (Q2.2.2)?

A2.2. Eq. (A2.1.2) and

$$\frac{\partial f}{\partial z^*} = \frac{1}{2}\left(\frac{\partial \operatorname{Re} f}{\partial x} - \frac{\partial \operatorname{Im} f}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial \operatorname{Im} f}{\partial x} + \frac{\partial \operatorname{Re} f}{\partial y}\right) \tag{A2.2.1}$$

give

$$\frac{\partial f}{\partial z}\frac{\partial f}{\partial z^*} = \frac{1}{4}\left[(\nabla \operatorname{Re} f)^2 - (\nabla \operatorname{Im} f)^2\right] + \frac{i}{2}\left[(\nabla \operatorname{Re} f) \cdot (\nabla \operatorname{Im} f)\right] \tag{A2.2.2}$$

and

$$\left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial z^*}\right|^2 = \frac{1}{2}(\nabla \operatorname{Re} f) \wedge (\nabla \operatorname{Im} f) \tag{A2.2.3}$$

Eq. (Q2.2.1) means that the contours of constant $\operatorname{Re} f$ and $\operatorname{Im} f$ are orthogonal and equally spaced. Therefore the mapping $f : \mathbb{C} \to \mathbb{C}$ preserves angles. Eq. (Q2.2.2) is the determinant of the mapping $\boldsymbol{dz} \mapsto \boldsymbol{df}$. Its sign means that holomorphic functions preserve orientation, while anti-holomorphic functions reverse orientation. A mapping that preserves angles and orientation is a conformal mapping.

We can also see this using Eq. (2.1.5) in the lecture notes. If f is holomorphic, Eq. (2.1.5) reduces to

$$df = \frac{\partial f}{\partial z}\, dz \tag{A2.2.4}$$

in which case the mapping $\boldsymbol{dz} \mapsto \boldsymbol{df}$ is the product of isotropic scaling by a factor $|\partial f/\partial z|$ and rotation by an angle $\arg(\partial f/\partial z)$. If f is anti-holomorphic, Eq. (2.1.5) reduces to

$$df = \frac{\partial f}{\partial z^*}\, dz^* \tag{A2.2.5}$$

in which case the mapping $\boldsymbol{dz} \mapsto \boldsymbol{df}$ is the product of the reflection $dz \mapsto dz^*$, isotropic scaling by a factor $|\partial f/\partial z^*|$ and rotation by an angle $\arg(\partial f/\partial z^*)$.

Q2.3. Consider the coordinates $w = u + iv$ defined relative to the Cartesian coordinates $z = x + iy$ by the holomorphic function

$$z = f(w) \tag{Q2.3.1}$$

(a) What properties do the coordinates (u, v) have?

(b) Express the Laplacian in terms of w and hence u and v.

(c) Use PGF to draw the coordinates (u, v) on the (x, y) plane in the case

$$z = \cosh w \tag{Q2.3.2}$$

A2.3. (a) Eq. (Q2.3.1) is a holomorphic, and hence conformal, mapping and so preserves the orthogonality of the Cartesian coordinate system. Hence the coordinates (u, v) will be orthogonal. See Question 2.2.

(b)

$$\nabla^2 = 4\frac{\partial^2}{\partial z \partial z^*} = \frac{4}{f' f'^*} \frac{\partial^2}{\partial w \partial w^*} = \frac{1}{|f'|^2} \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \tag{A2.3.1}$$

(c)

$$x = \cosh u \cos v \tag{A2.3.2}$$

$$y = \sinh u \sin v \tag{A2.3.3}$$

(u, v) are a kind of generalized polar coordinates with

$$u \in [0, \infty) \tag{A2.3.4}$$

$$v \in (-\pi, \pi] \tag{A2.3.5}$$

Eliminating v gives

$$\frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = 1 \tag{A2.3.6}$$

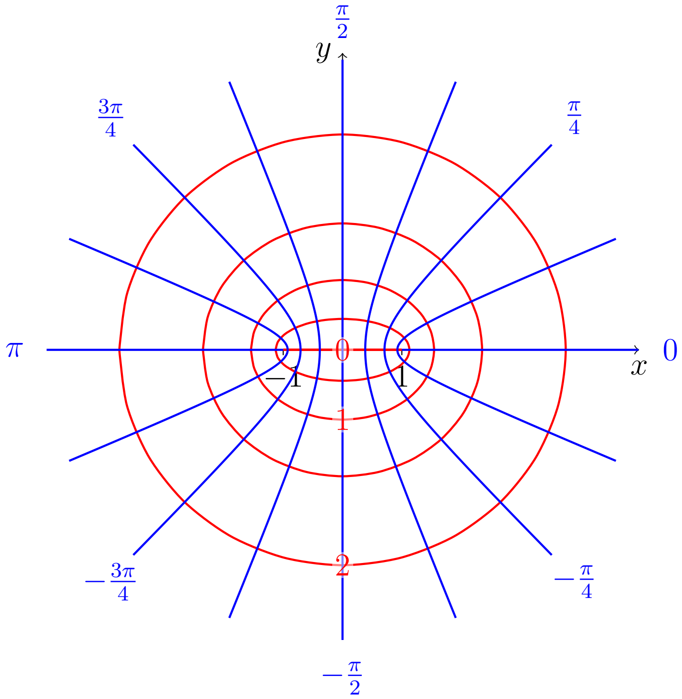

Figure A2.3.1: Contours of constant u and v.

and hence the contours of constant u are ellipses, with $u = 0$ corresponding to the line segment $-1 \leq x \leq 1$ and $y = 0$, and asymptoting to the polar coordinate $r = e^u/2$ circles at large u.

Eliminating u gives

$$\frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = 1 \tag{A2.3.7}$$

and hence the contours of constant v are hyperbolae, asymptoting to the polar coordinate $\theta = v$ radial lines at large u.