Homework 3

Q3.1. The superpotential W of the Minimal Supersymmetric Standard Model is a holomorphic function of Q, u, d, L, e, H_u and H_d that is invariant under the U(1) hypercharge symmetry

$$\begin{array}{llllllllllll} Q & \to & e^{i\theta}Q & , & u & \to & e^{-4i\theta}u & , & d & \to & e^{2i\theta}d & , & L & \to & e^{-3i\theta}L \\ e & \to & e^{6i\theta}e & , & H_u & \to & e^{3i\theta}H_u & , & H_d & \to & e^{-3i\theta}H_d \end{array} \tag{Q3.1.1}$$

and the $\mathbb{Z}_2$ R-parity symmetry

$$\begin{array}{llllllllllll} Q & \to & -Q & , & u & \to & -u & , & d & \to & -d & , & L & \to & -L \\ e & \to & -e & , & H_u & \to & H_u & , & H_d & \to & H_d \end{array} \tag{Q3.1.2}$$

Expand W in a Taylor series and determine the terms up to cubic order.

A3.1. Expanding $W(Q, u, d, L, e, H_u, H_d)$ in a Taylor series, the first term is a constant. There are no linear terms invariant under hypercharge symmetry, since all fields are charged. The only invariant quadratic term is proportional to H_uH_d, since H_u and H_d are the only fields with opposite hypercharge and the same R-parity. The cubic terms must contain an odd number of Higgs fields (H_u or H_d) to be invariant under R-parity, but three Higgs fields would not be invariant under hypercharge, so there must be exactly one Higgs field in each cubic term. The Higgs fields have hypercharge ± 3 so for invariance the non-Higgs fields' charge must add to ∓ 3, leaving the three allowed cubic terms QH_uu, QH_dd and LH_de. Therefore, up to cubic order

$$W = W_0 + \mu H_u H_d + \lambda_u Q H_u u + \lambda_d Q H_d d + \lambda_e L H_d e \tag{A3.1.1}$$

where W_0, μ, λ_u, λ_d and λ_e are constants.

Eq. (A3.1.1) is the superpotential of the Minimal Supersymmetric Standard Model. If it is correct, it should be discovered soon by the Large Hadron Collider. H_u and H_d are Higgs superfields and μ contributes to their mass. The Higgs fields have non-zero values giving mass to the quarks and electron via the cubic terms in W. The higher order terms, though much weaker for small field values, can also give rise to observable effects. For example, the quartic term $(LH_u)^2$ is thought to be the origin of neutrino masses. Other higher order terms violate the accidental baryon symmetry

$$Q \to e^{i\phi}Q \quad , \quad u \to e^{-i\phi}u \quad , \quad d \to e^{-i\phi}d \tag{A3.1.2}$$

and may cause proton decay. Even the constant term is important, giving mass to the gravitino component of the graviton superfield and contributing to the vacuum energy.

Q3.2. Calculate

$$\sum_{n=0}^{\infty} x^n \tag{Q3.2.1}$$

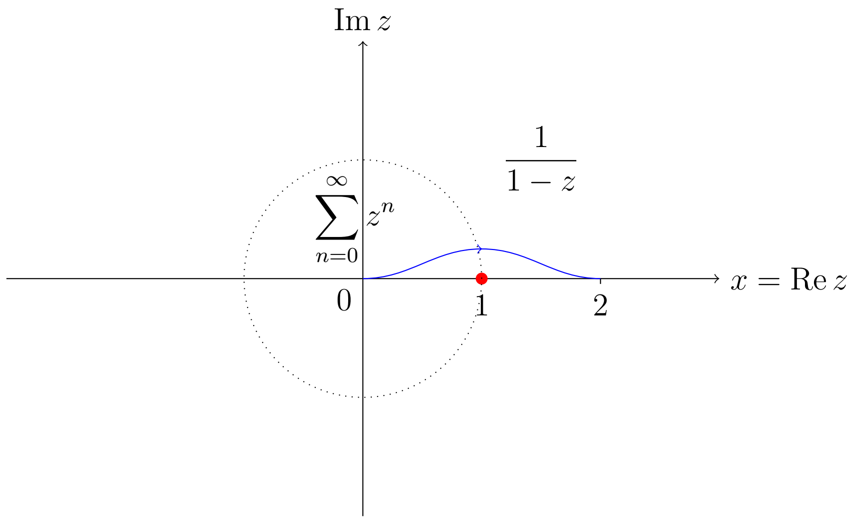

Figure A3.2.1: Analytic continuation around the singularity at $z = 1$.

for $x = 2$.

Use PGF to draw a diagram illustrating your answer.

A3.2. The sum in Eq. (Q3.2.1) converges for $|x| < 1$ but diverges for $|x| > 1$. Thus it describes a function for $|x| < 1$ but for $|x| > 1$ we need a different description of the function. In particular

$$\left. \sum_{n=0}^{\infty} x^n \right|_{x=2} \neq \sum_{n=0}^{\infty} 2^n \tag{A3.2.1}$$

The real function

$$f(x) = \sum_{n=0}^{\infty} x^n \qquad \text{for } |x| < 1 \tag{A3.2.2}$$

can be analytically continued to $x \leq -1$ but the singularity at $x = 1$ forms an impenetrable barrier to analytic continuation to $x > 1$.

However, $f(x)$ can be uniquely extended to the holomorphic function

$$f(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \qquad \text{for } |z| < 1 \tag{A3.2.3}$$

which can be analytically continued around the singularity at $z = 1$, giving

$$f(2) = -1 \tag{A3.2.4}$$

See Figure A3.2.1.