Homework 4

Optional extra question.

Q4.3. Let $f(z, z^*)$ be a complex function and $\boldsymbol{f}$ be the vector field on the complex plane with Cartesian components $(\operatorname{Re} f, \operatorname{Im} f)$. Show that

$$\int_{\partial A} f \, dz^* = \int_A (\boldsymbol{\nabla} \wedge \boldsymbol{f} - i \boldsymbol{\nabla} \cdot \boldsymbol{f}) \, dA \tag{Q4.3.1}$$

where ∂A is the boundary of the area A in the complex plane. Interpret

$$f = \frac{1}{2\pi z^*} \tag{Q4.3.2}$$

in terms of two dimensional electromagnetism.

A4.3. The first part of this question can be answered in at least two ways:

(a)

$$\int_{\partial A} f \, dz^* = \int_A \frac{\partial f}{\partial z} \, dz \, dz^* \tag{A4.3.1}$$

Now using Eq. (A2.1.3)

$$\frac{\partial f}{\partial z} = \frac{1}{2} \boldsymbol{\nabla} \cdot \boldsymbol{f} + \frac{i}{2} \boldsymbol{\nabla} \wedge \boldsymbol{f} \tag{A4.3.2}$$

and

$$dz \, dz^* = -2i \, dx \, dy \tag{A4.3.3}$$

gives Eq. (Q4.3.1).

(b)

$$\int_{\partial A} f \, dz^* = \int_{\partial A} (\boldsymbol{f} \cdot \boldsymbol{dz} - i \boldsymbol{f} \cdot \boldsymbol{dn}) \tag{A4.3.4}$$

where $\boldsymbol{dz}$ has components $(\operatorname{Re} dz, \operatorname{Im} dz)$ and $\boldsymbol{dn}$ has components $(\operatorname{Im} dz, -\operatorname{Re} dz)$. Now using Stokes' and Gauss' theorems gives Eq. (Q4.3.1).

Now

$$\int_{\partial A} \frac{1}{2\pi z^*} \, dz^* = -i \tag{A4.3.5}$$

for any boundary ∂A going anticlockwise once around the origin. Therefore Eq. (Q4.3.1) gives

$$\int_A \boldsymbol{\nabla} \wedge \boldsymbol{f} \, dA = 0 \tag{A4.3.6}$$

$$\int_A \boldsymbol{\nabla} \cdot \boldsymbol{f} \, dA = 1 \tag{A4.3.7}$$

Therefore we can interpret the $\boldsymbol{f}$ corresponding to Eq. (Q4.3.2)

$$\boldsymbol{f} = \frac{\boldsymbol{z}}{2\pi |z|^2} \tag{A4.3.8}$$

as the electric field of a unit charge at the origin.