Homework 4

Q4.1. Calculate

$$\int_0^\infty \frac{\sin(px)}{x}\,dx \tag{Q4.1.1}$$

for $p \in \mathbb{R}$.

Use PGF to draw a diagram illustrating your answer.

A4.1. Let

$$I(p) = \int_0^\infty \frac{\sin(px)}{x}\,dx \tag{A4.1.1}$$

then

$$I(p) = \left\{ \begin{array}{ccc} I(1) & \text{for} & p > 0 \\ 0 & \text{for} & p = 0 \\ -I(1) & \text{for} & p < 0 \end{array} \right\} = I(1)\,\mathrm{sgn}\,p \tag{A4.1.2}$$

Now

$$\begin{aligned} I(1) &= \int_0^\infty \frac{\sin x}{x}\,dx && \text{(A4.1.3)} \\ &= \frac{1}{2}\int_{-\infty}^\infty \frac{\sin x}{x}\,dx && \text{(A4.1.4)} \\ &= \frac{1}{2}\int_C \frac{\sin z}{z}\,dz \qquad \left(\neq \frac{1}{2}\,\mathrm{Im}\int_C \frac{e^{iz}}{z}\,dz \ \text{ for } C \not\subset \mathbb{R}\right) && \text{(A4.1.5)} \\ &= \frac{1}{4i}\int_C \frac{e^{iz}}{z}\,dz - \frac{1}{4i}\int_C \frac{e^{-iz}}{z}\,dz && \text{(A4.1.6)} \end{aligned}$$

where we can take the contour C to pass either above or below the origin, but must be consistent in our choice. We take it to pass below the origin, see Figure A4.1.1.

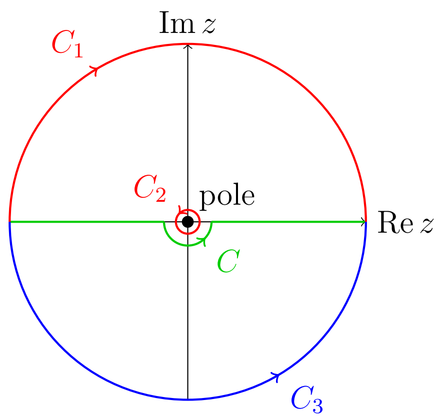

Figure A4.1.1: Contour integration of $\int_{-\infty}^{\infty} \frac{\sin z}{z}\,dz$.

Deforming the contour towards $i\infty$ for the first integral in Eq. (A4.1.6), the part of the contour at infinity vanishes due to the e^{iz} factor, leaving the part of the contour wrapped anticlockwise around the pole at the origin

$$\frac{1}{4i}\int_C \frac{e^{iz}}{z}\,dz = \frac{1}{4i}\int_{C_1} \frac{e^{iz}}{z}\,dz + \frac{1}{4i}\int_{C_2} \frac{e^{iz}}{z}\,dz = 0 + \frac{\pi}{2} \tag{A4.1.7}$$

Deforming the contour towards $-i\infty$ for the second integral in Eq. (A4.1.6) gives zero due to the e^{-iz} factor

$$\frac{1}{4i}\int_C \frac{e^{-iz}}{z}\,dz = \frac{1}{4i}\int_{C_3} \frac{e^{-iz}}{z}\,dz = 0 \tag{A4.1.8}$$

Putting it all together

$$\int_0^\infty \frac{\sin(px)}{x}\,dx = \frac{\pi}{2}\,\mathrm{sgn}\,p \tag{A4.1.9}$$

Q4.2. Calculate

$$\int_0^1 \frac{da}{z-a} \tag{Q4.2.1}$$

Compare the holomorphic structure of Eq. (Q4.2.1) with that of your answer.

Calculate

$$\int_C dz \int_0^1 \frac{da}{z-a} \tag{Q4.2.2}$$

where C is an anticlockwise loop around 0 and 1. Check your answer by reversing the order of integration.

Use PGF to draw a diagram illustrating your answer.

A4.2.

$$\int_0^1 \frac{da}{z-a} = \ln\left(\frac{z}{z-1}\right) \tag{A4.2.1}$$

The left hand side is a sum of poles along a curve from 0 to 1, while the right hand side has branch points at 0 and 1 with a branch cut along the curve from 0 to 1.

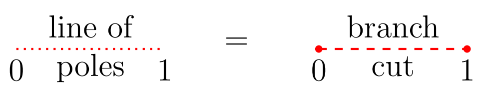

Figure A4.2.1: Holomorphic structure of Eq. (A4.2.1).

$\ln\left(\frac{z}{z-1}\right)$ is analytic outside the branch cut, and for large $|z|$

$$\ln\left(\frac{z}{z-1}\right) = -\ln\left(1-\frac{1}{z}\right) \sim \frac{1}{z} + \ldots \tag{A4.2.2}$$

Therefore, deforming C to large $|z|$,

$$\int_C dz \int_0^1 \frac{da}{z-a} = \int_C dz \ln\left(\frac{z}{z-1}\right) = 2\pi i \tag{A4.2.3}$$

in agreement with

$$\int_0^1 da \int_C \frac{dz}{z-a} = \int_0^1 da \, 2\pi i = 2\pi i \tag{A4.2.4}$$