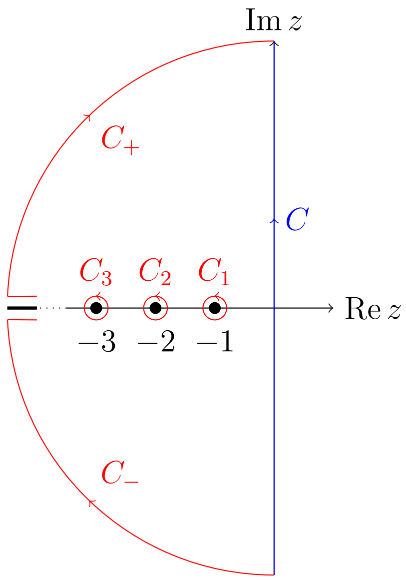

Figure A5.4.1: Contour integration of $\int_{-i\infty}^{i\infty} z!\,dz$.

Homework 5

Optional extra question.

Q5.4. Use contour integration to determine

$$\int_{-i\infty}^{i\infty} z!\,dz \tag{Q5.4.1}$$

A5.4. Using Section 2.3.1, $z! = \Gamma(z+1)$ has simple poles at the negative integers

$$\begin{aligned} z! &= \frac{(z+n)!}{(z+1)\cdots(z+n-1)(z+n)} && \text{(A5.4.1)} \\ &= \frac{(-1)^{n-1}(z+n)!}{[n-1-(z+n)]\cdots[1-(z+n)](z+n)} && \text{(A5.4.2)} \\ &\sim \frac{(-1)^{n-1}}{(n-1)!\,(z+n)} \qquad \text{for } z \simeq -n && \text{(A5.4.3)} \end{aligned}$$

and, using Section 2.3.2, the asymptotic behaviour

$$\begin{aligned} z! &= z\,\Gamma(z) \sim \sqrt{2\pi}\,z^{\frac{1}{2}} \exp\{z(\ln z - 1)\} && \text{(A5.4.4)} \\ &\sim \sqrt{2\pi}\,z^{\frac{1}{2}} \exp\{i[\operatorname{Im} z(\ln|z|-1) + \operatorname{Re} z \arg z]\} \\ &\quad \times \exp\{[\operatorname{Re} z(\ln|z|-1) - \operatorname{Im} z \arg z]\} && \text{(A5.4.5)} \end{aligned}$$

with branch cut along the negative real axis. Therefore, deforming the contour as shown in Figure A5.4.1,

$$\int_{-i\infty}^{i\infty} z!\,dz = \int_{C_-} z!\,dz + \sum_{n=1}^{\infty} \int_{C_n} z!\,dz + \int_{C_+} z!\,dz = 0 + 2\pi i \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} + 0 = \frac{2\pi i}{e} \tag{A5.4.6}$$