Homework 5

Q5.1.

$$I(x) = \int_{-\pi}^{\pi} e^{ix \sin t} dt \tag{Q5.1.1}$$

Use the transformation

$$z = e^{it} \tag{Q5.1.2}$$

to express $I(x)$ as a holomorphic integral over a closed curve. Then use

(a) contour integration to determine $I(x)$,

(b) the saddle point approximation to obtain the asymptotic form of $I(x)$ as $x \to \infty$.

A5.1. Substituting Eq. (Q5.1.2) into Eq. (Q5.1.1) gives

$$I(x) = \int_C \exp\left[\frac{1}{2}x\left(z - \frac{1}{z}\right)\right] \frac{dz}{iz} \tag{A5.1.1}$$

where the closed curve C is shown in Figure A5.1.1.

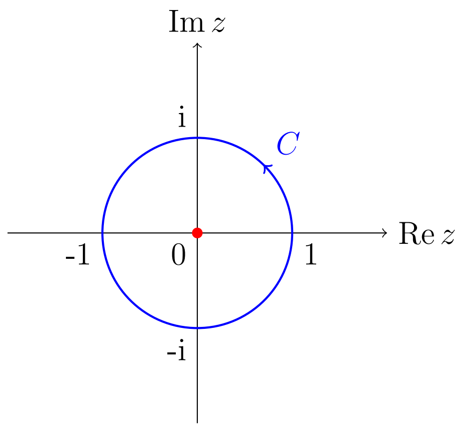

Figure A5.1.1: The curve C, and the singularity of the integrand, in Eq. (A5.1.1).

(a) Since C is closed, we only need to determine the simple poles of the integrand. Expanding in Laurent series about the essential singularity at the origin

$$I(x) = \int_C \exp\left(\frac{xz}{2}\right) \exp\left(-\frac{x}{2z}\right) \frac{dz}{iz} \tag{A5.1.2}$$

$$= \int_C \left[\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{xz}{2}\right)^n\right] \left[\sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\left(\frac{x}{2z}\right)^m\right] \frac{dz}{iz} \tag{A5.1.3}$$

The simple poles occur for $m = n$, other terms giving zero, therefore

$$I(x) = \int_C \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n} \frac{dz}{iz} \tag{A5.1.4}$$

$$= 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n} \qquad [= 2\pi\, J_0(x)] \tag{A5.1.5}$$

(b) The coefficient of the exponent of x in Eq. (A5.1.1) is

$$f(z) = \frac{1}{2}\left(z - \frac{1}{z}\right) \tag{A5.1.6}$$

and

$$f'(z) = \frac{1}{2}\left(1 + \frac{1}{z^2}\right) \tag{A5.1.7}$$

$$f''(z) = -\frac{1}{z^3} \tag{A5.1.8}$$

so $f(z)$ has saddle points at

$$z = \pm i \tag{A5.1.9}$$

with

$$f(\pm i + \delta z) = \pm i \mp \frac{i}{2}(\delta z)^2 + \ldots \tag{A5.1.10}$$

Deforming the curve C to pass over the saddle points $z = \pm i$ along paths of steepest descent, as illustrated in Figure A5.1.2, and taking the limit $x \to \infty$,

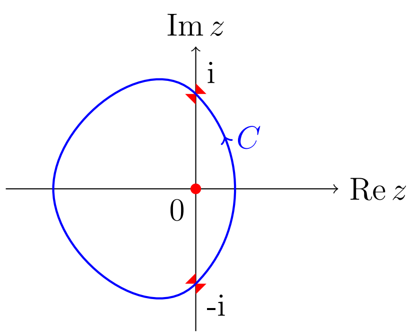

Figure A5.1.2: C deformed to pass over the saddles along paths of steepest descent ($x > 0$).

Eqs. (A5.1.1) and (A5.1.10) give

$$I(x) \overset{x\to\infty}{\sim} -\int_{C\simeq i} \exp\left\{x\left[i - \frac{i}{2}(z-i)^2 + \ldots\right]\right\} dz + \int_{C\simeq -i} \exp\left\{x\left[-i + \frac{i}{2}(z+i)^2 + \ldots\right]\right\} dz \tag{A5.1.11}$$

$$\overset{x\to\infty}{\sim} \quad -e^{\frac{3i\pi}{4}} e^{ix} \int_{-\infty}^{\infty} e^{-\frac{x}{2}s^2}\, ds + e^{\frac{i\pi}{4}} e^{-ix} \int_{-\infty}^{\infty} e^{-\frac{x}{2}s^2}\, ds \qquad \text{(A5.1.12)}$$

$$= \sqrt{\frac{2\pi}{x}} \left[e^{i\left(x-\frac{\pi}{4}\right)} + e^{-i\left(x-\frac{\pi}{4}\right)} \right] \qquad \text{(A5.1.13)}$$

Q5.2. In the path integral formulation of quantum mechanics, the amplitude for a particle moving from $x_{\rm i}(t_{\rm i})$ to $x_{\rm f}(t_{\rm f})$ is given by

$$\int d[x(t)] \exp\left\{ \frac{i}{\hbar} S[x(t)] \right\} \qquad \text{(Q5.2.1)}$$

where the integral is over all paths $x(t)$ going from $x_{\rm i}(t_{\rm i})$ to $x_{\rm f}(t_{\rm f})$. Derive classical physics.

A5.2. Classical physics corresponds to the limit $\hbar \to 0$. In this limit the coefficient of the exponent becomes very large. Contributions to the integral will then tend to cancel due to the rapidly rotating phase except at points where $S[x(t)]$ is stationary. Thus in the classical limit we expect the integral to be dominated by paths with

$$\delta S = 0 \qquad \text{(A5.2.1)}$$

Q5.3. Calculate

$$\sum_{n=0}^{\infty} \frac{n!}{x^n} \qquad \text{(Q5.3.1)}$$

for $x = 10$.

Use PGF to draw a diagram illustrating your answer.

A5.3. Eq. (Q5.3.1) is an asymptotic series (see Section 2.3.3). It does not converge but nevertheless can give accurate results for large values of x if the series is truncated before it starts to diverge. For $x = 10$ the series starts to diverge between $n = 9$ and $n = 10$ and so we take

$$\left. \sum_{n=0}^{\infty} \frac{n!}{x^n} \right|_{x=10} \simeq \sum_{n=0}^{9} \frac{n!}{10^n} \qquad \text{(A5.3.1)}$$

$$= 1 + 0.1 + 0.02 + 0.006 + 0.0024 + 0.0012 + 0.00072 + 0.000504 + 0.0004032 + 0.00036288 \qquad \text{(A5.3.2)}$$

$$\simeq 1.1316 \qquad \text{(A5.3.3)}$$

compared with

$$\left. x e^{-x}\, \mathrm{Ei}(x) \right|_{x=10} \simeq 1.1315 \qquad \text{(A5.3.4)}$$

which has Eq. (Q5.3.1) as its asymptotic series. See Figure A5.3.1.

Note that $\mathrm{Ei}(x)$ also has the convergent series

$$\mathrm{Ei}(x) = \gamma + \ln x + \sum_{n=1}^{\infty} \frac{x^n}{n n!} \qquad \text{(A5.3.5)}$$

where $\gamma \simeq 0.577216$ is the Euler-Mascheroni constant, but this series only starts to converge when $n \sim x$ and would require many more terms to give an accurate answer. See Figure A5.3.1.

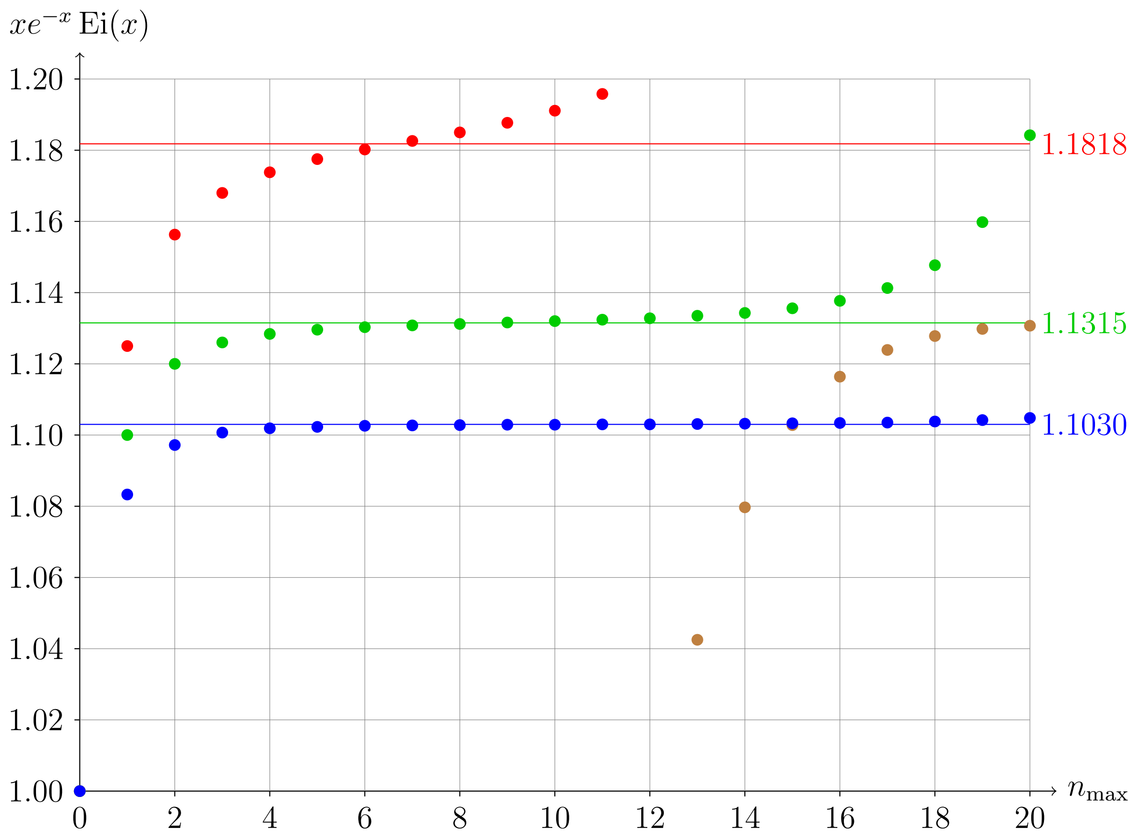

Figure A5.3.1: Convergence of the asymptotic series Eq. (Q5.3.1) for $x = 8, 10, 12$. The exact values are given on the right hand side. Also shown is the convergence of the convergent series Eq. (A5.3.5) for $x = 10$.