Homework 6

Q6.1. Consider the set of functions satisfying the boundary condition

$$\phi(a) = \phi(b) = \phi_0 \tag{Q6.1.1}$$

Does it form a vector space?

A6.1. It only forms a vector space if $\phi_0 = 0$ since otherwise the space is not closed under addition or scalar multiplication.

Q6.2. Show that

(a)

$$A^{\dagger\dagger} = A \tag{Q6.2.1}$$

(b)

$$(AB)^\dagger = B^\dagger A^\dagger \tag{Q6.2.2}$$

(c)

$$\left(A^\dagger\right)^{-1} = \left(A^{-1}\right)^\dagger \tag{Q6.2.3}$$

A6.2. (a)

$$\langle\phi| A^{\dagger\dagger} |\psi\rangle = \left(\langle\psi| A^\dagger |\phi\rangle\right)^* = \left(\langle\phi| A |\psi\rangle\right)^{**} = \langle\phi| A |\psi\rangle \tag{A6.2.1}$$

(b)

$$\langle\phi| (AB)^\dagger |\psi\rangle = \left(\langle\psi| AB |\phi\rangle\right)^\dagger = \left(B |\phi\rangle\right)^\dagger \left(\langle\psi| A\right)^\dagger = \langle\phi| B^\dagger A^\dagger |\psi\rangle \tag{A6.2.2}$$

(c)

$$\left(A^{-1}\right)^\dagger A^\dagger = \left(AA^{-1}\right)^\dagger = 1^\dagger = 1 = 1^\dagger = \left(A^{-1}A\right)^\dagger = A^\dagger \left(A^{-1}\right)^\dagger \tag{A6.2.3}$$

Q6.3. Show that

(a) $AA^\dagger$ is Hermitian,

(b) e^{iH} is unitary,

(c) $U_1 U_2$ is unitary.

where A is a linear operator, H is a Hermitian operator, and U_1 and U_2 are unitary operators.

A6.3. (a)

$$\left(AA^\dagger\right)^\dagger = A^{\dagger\dagger} A^\dagger = AA^\dagger \tag{A6.3.1}$$

(b)

$$\left(e^{iH}\right)^\dagger = e^{i^* H^\dagger} = e^{-iH} = \left(e^{iH}\right)^{-1} \tag{A6.3.2}$$

(c)

$$(U_1U_2)^\dagger = U_2^\dagger U_1^\dagger = U_2^{-1}U_1^{-1} = (U_1U_2)^{-1} \tag{A6.3.3}$$

Q6.4. Describe the properties of

$$\frac{|\phi\rangle\langle\psi|}{\langle\psi|\phi\rangle} \tag{Q6.4.1}$$

A6.4.

$$P \equiv \frac{|\phi\rangle\langle\psi|}{\langle\psi|\phi\rangle} \tag{A6.4.1}$$

is a linear operator.

(a) If $\langle\psi|\phi\rangle = 0$ then P is divergent.

(b) If $|\psi\rangle \propto |\phi\rangle$ then

$$P = \frac{|\phi\rangle\langle\phi|}{\langle\phi|\phi\rangle} \tag{A6.4.2}$$

with

$$P^2 = P \tag{A6.4.3}$$

and

$$P^\dagger = P \tag{A6.4.4}$$

and so P is the projection operator that projects onto the subspace generated by $|\phi\rangle$ (or equivalently $|\psi\rangle$).

(c) If $\langle\psi|\phi\rangle \neq 0$ and $|\psi\rangle \not\propto |\phi\rangle$ then

$$P^2 = P \tag{A6.4.5}$$

but

$$P^\dagger \neq P \tag{A6.4.6}$$

and so P is a non-Hermitian projection operator. It projects orthogonally to the subspace generated by $|\psi\rangle$ onto the subspace generated by $|\phi\rangle$, see Figure A6.4.1.

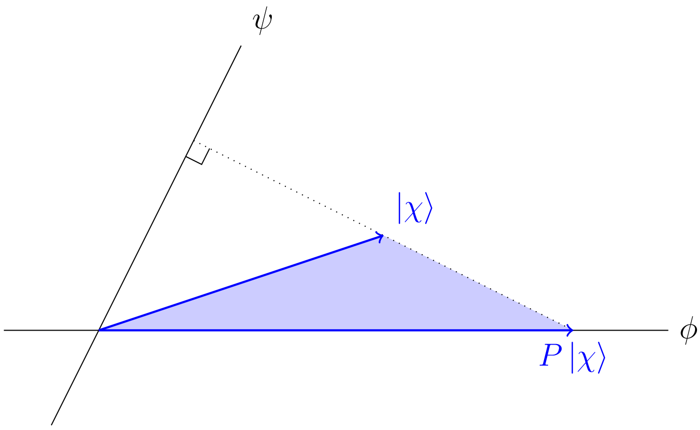

Figure A6.4.1: Projection by the non-Hermitian projection operator $P \equiv \frac{|\phi\rangle\langle\psi|}{\langle\psi|\phi\rangle}$.