Homework 7

Q7.1. Show that the eigenvalues α of a

(a) Hermitian operator satisfy

$$\alpha^* = \alpha \tag{Q7.1.1}$$

(b) unitary operator satisfy

$$\alpha^* = \alpha^{-1} \tag{Q7.1.2}$$

A7.1. (a) Let

$$H^\dagger = H \tag{A7.1.1}$$

and

$$H P_{\mathcal{H}_\alpha} = \alpha P_{\mathcal{H}_\alpha} \tag{A7.1.2}$$

then

$$\alpha^* P^\dagger_{\mathcal{H}_\alpha} P_{\mathcal{H}_\alpha} = \left(P^\dagger_{\mathcal{H}_\alpha} H P_{\mathcal{H}_\alpha} \right)^\dagger = P^\dagger_{\mathcal{H}_\alpha} H^\dagger P_{\mathcal{H}_\alpha} = P^\dagger_{\mathcal{H}_\alpha} H P_{\mathcal{H}_\alpha} = \alpha P^\dagger_{\mathcal{H}_\alpha} P_{\mathcal{H}_\alpha} \tag{A7.1.3}$$

(b) Let

$$U^\dagger = U^{-1} \tag{A7.1.4}$$

and

$$U P_{\mathcal{U}_\alpha} = \alpha P_{\mathcal{U}_\alpha} \tag{A7.1.5}$$

with $P_{\mathcal{U}_\alpha} \neq 0$, then $\alpha \neq 0$,

$$U^{-1} P_{\mathcal{U}_\alpha} = \alpha^{-1} P_{\mathcal{U}_\alpha} \tag{A7.1.6}$$

and

$$\alpha^* P^\dagger_{\mathcal{U}_\alpha} P_{\mathcal{U}_\alpha} = \left(P^\dagger_{\mathcal{U}_\alpha} U P_{\mathcal{U}_\alpha} \right)^\dagger = P^\dagger_{\mathcal{U}_\alpha} U^\dagger P_{\mathcal{U}_\alpha} = P^\dagger_{\mathcal{U}_\alpha} U^{-1} P_{\mathcal{U}_\alpha} = \alpha^{-1} P^\dagger_{\mathcal{U}_\alpha} P_{\mathcal{U}_\alpha} \tag{A7.1.7}$$

Q7.2. Show that the eigenspaces of a

(a) Hermitian operator

(b) unitary operator

are orthogonal.

A7.2. (a) Let

$$H^\dagger = H \tag{A7.2.1}$$

and

$$H P_{\mathcal{H}_\alpha} = \alpha P_{\mathcal{H}_\alpha} \tag{A7.2.2}$$

then using Eq. (Q7.1.1)

$$\alpha P_{\mathcal{H}_\alpha}^\dagger P_{\mathcal{H}_\beta} = \alpha^* P_{\mathcal{H}_\alpha}^\dagger P_{\mathcal{H}_\beta} = \left(P_{\mathcal{H}_\beta}^\dagger H P_{\mathcal{H}_\alpha} \right)^\dagger = P_{\mathcal{H}_\alpha}^\dagger H P_{\mathcal{H}_\beta} = \beta P_{\mathcal{H}_\alpha}^\dagger P_{\mathcal{H}_\beta} \quad \text{(A7.2.3)}$$

therefore

$$P_{\mathcal{H}_\alpha}^\dagger P_{\mathcal{H}_\beta} = 0 \qquad \text{for } \alpha \neq \beta \quad \text{(A7.2.4)}$$

(b) Let

$$U^\dagger = U^{-1} \quad \text{(A7.2.5)}$$

and

$$U P_{\mathcal{U}_\alpha} = \alpha P_{\mathcal{U}_\alpha} \quad \text{(A7.2.6)}$$

then using Eqs. (Q7.1.2) and (A7.1.6)

$$\alpha^{-1} P_{\mathcal{U}_\alpha}^\dagger P_{\mathcal{U}_\beta} = \alpha^* P_{\mathcal{U}_\alpha}^\dagger P_{\mathcal{U}_\beta} = \left(P_{\mathcal{U}_\beta}^\dagger U P_{\mathcal{U}_\alpha} \right)^\dagger = P_{\mathcal{U}_\alpha}^\dagger U^{-1} P_{\mathcal{U}_\beta} = \beta^{-1} P_{\mathcal{U}_\alpha}^\dagger P_{\mathcal{U}_\beta} \quad \text{(A7.2.7)}$$

therefore

$$P_{\mathcal{U}_\alpha}^\dagger P_{\mathcal{U}_\beta} = 0 \qquad \text{for } \alpha \neq \beta \quad \text{(A7.2.8)}$$

Q7.3. The position and momentum operators have the commutation relation

$$[\hat{x}, \hat{p}] = i\hbar \quad \text{(Q7.3.1)}$$

Show that they cannot have a common eigenvector. Interpret physically.

A7.3. Suppose

$$\hat{x} \, |x, p\rangle = x \, |x, p\rangle \quad \text{(A7.3.1)}$$

and

$$\hat{p} \, |x, p\rangle = p \, |x, p\rangle \quad \text{(A7.3.2)}$$

then

$$[\hat{x}, \hat{p}] \, |x, p\rangle = (xp - px) \, |x, p\rangle = 0 \quad \text{(A7.3.3)}$$

contradicting Eq. (Q7.3.1).

An eigenvector of $\hat{x}$ is a state with definite position and an eigenvector of $\hat{p}$ is a state with definite momentum. Thus Eq. (Q7.3.1) implies that a quantum particle cannot have definite position and momentum at the same time.