Homework 8

Q8.1. Show that

(a)
$$\delta(-x) = \delta(x) \tag{Q8.1.1}$$

(b)
$$\delta'(-x) = -\,\delta'(x) \tag{Q8.1.2}$$

(c)
$$x\,\delta(x) = 0 \tag{Q8.1.3}$$

(d)
$$x\,\delta'(x) = -\,\delta(x) \tag{Q8.1.4}$$

A8.1. (a) Using Eq. (3.3.18),

$$\int_{-\infty}^{\infty} dy\,\delta(-x+y)\,f(y) = \int_{-\infty}^{\infty} dy\,\delta(-x-y)\,f(-y) = f(x) = \int_{-\infty}^{\infty} dy\,\delta(x-y)\,f(y) \tag{A8.1.1}$$

hence Eq. (Q8.1.1).

(b) Taking the derivative of Eq. (Q8.1.1) gives Eq. (Q8.1.2).

(c)
$$\int_{-\infty}^{\infty} dx\,x\,\delta(x)\,f(x) = 0 \times f(0) = 0 \tag{A8.1.2}$$

hence Eq. (Q8.1.3).

(d) Taking the derivative of Eq. (Q8.1.3) gives Eq. (Q8.1.4).

Q8.2. Show that

(a)
$$\delta(x) = \theta'(x) \tag{Q8.2.1}$$

where
$$\theta(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 1 & \text{for} \quad x > 0 \end{cases} \tag{Q8.2.2}$$

(b)
$$\int_{-\infty}^{\infty} dy\,\delta'(x-y)\,f(y) = f'(x) \tag{Q8.2.3}$$

A8.2. (a) Integrating by parts

$$\begin{aligned} \int_{-\infty}^{\infty} dx\,\theta'(x)\,f(x) &= [\theta(x)\,f(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx\,\theta(x)\,f'(x) && \text{(A8.2.1)} \\ &= f(\infty) - \int_{0}^{\infty} dx\,f'(x) && \text{(A8.2.2)} \\ &= f(0) && \text{(A8.2.3)} \end{aligned}$$

hence Eq. (Q8.2.1).

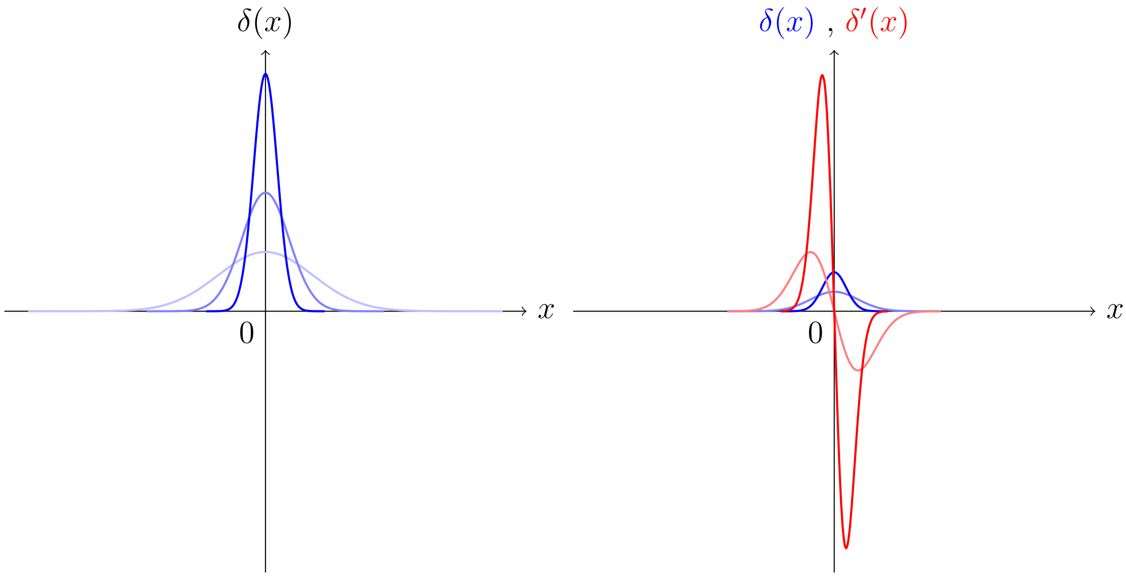
(b) Integrating by parts

$$\int_{-\infty}^{\infty} dy\, \delta'(x-y)\, f(y) = [-\,\delta(x-y)\, f(y)]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} dy\, \delta(x-y)\, f'(y) \tag{A8.2.4}$$

$$= f'(x) \tag{A8.2.5}$$

Q8.3. Use PGF to draw diagrams illustrating $\delta(x)$ and $\delta'(x)$.

A8.3.



Q8.4. Show that

(a)

$$\delta(x) = \lim_{\epsilon \to 0^+} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ikx - \epsilon k^2} \tag{Q8.4.1}$$

(b)

$$\delta(x) = \lim_{\Lambda \to \infty} \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} dk\, e^{ikx} \tag{Q8.4.2}$$

A8.4. (a) Define

$$\delta_\epsilon(x) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ikx - \epsilon k^2} \tag{A8.4.1}$$

$$= \frac{1}{2\pi} e^{-\frac{x^2}{4\epsilon}} \int_{-\infty}^{\infty} dk\, e^{-\epsilon \left(k - \frac{ix}{2\epsilon}\right)^2} \tag{A8.4.2}$$

$$= \frac{1}{2\pi} e^{-\frac{x^2}{4\epsilon}} \int_{-\infty}^{\infty} dk\, e^{-\epsilon k^2} \tag{A8.4.3}$$

$$= \frac{1}{\sqrt{4\pi\epsilon}} e^{-\frac{x^2}{4\epsilon}} \tag{A8.4.4}$$

Therefore

$$\lim_{\epsilon\to 0^+} \delta_\epsilon(x) = \begin{cases} \infty & \text{if} \quad x = 0 \\ 0 & \text{if} \quad x \neq 0 \end{cases} \tag{A8.4.5}$$

and $\delta_\epsilon(x)$ has a simple spike around $x = 0$ with

$$\int_{-\infty}^{\infty} dx\, \delta_\epsilon(x) = 1 \tag{A8.4.6}$$

Therefore

$$\lim_{\epsilon\to 0^+} \delta_\epsilon(x) = \delta(x) \tag{A8.4.7}$$

(b) Define

$$\delta_\Lambda(x) \equiv \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} dk\, e^{ikx} \tag{A8.4.8}$$

$$= \frac{\sin(\Lambda x)}{\pi x} \tag{A8.4.9}$$

Therefore

$$\int_{-\infty}^{\infty} dx\, \delta_\Lambda(x)\, f(x) = \int_{-\infty}^{\infty} dx\, \frac{\sin(\Lambda x)}{\pi x}\, f(x) \tag{A8.4.10}$$

$$= \int_{-\infty}^{\infty} dx\, \frac{\sin x}{\pi x}\, f\left(\frac{x}{\Lambda}\right) \tag{A8.4.11}$$

Therefore, using the results of Homework 4 Question 1,

$$\lim_{\Lambda\to\infty} \int_{-\infty}^{\infty} dx\, \delta_\Lambda(x)\, f(x) = \int_{-\infty}^{\infty} dx\, \frac{\sin x}{\pi x}\, f(0) \tag{A8.4.12}$$

$$= f(0) \tag{A8.4.13}$$

Therefore

$$\lim_{\Lambda\to\infty} \delta_\Lambda(x) = \delta(x) \tag{A8.4.14}$$