Homework 9

Optional extra question.

Q9.3. The Hermitian operators $\hat{x}$ and $\hat{p}$ satisfy

$$[\hat{x}, \hat{p}] = i \tag{Q9.3.1}$$

and

$$\hat{x}\,|x\rangle = x\,|x\rangle \tag{Q9.3.2}$$

Express $\hat{p}$ as a differential operator p_x. Calculate

$$e^{ia\hat{p}}\,|x\rangle \tag{Q9.3.3}$$

and

$$e^{iap_x}\,\phi(x) \tag{Q9.3.4}$$

A9.3. Taking the Hermitian conjugate of Eq. (Q9.3.2) gives

$$\langle x|\,\hat{x} = x\,\langle x| \tag{A9.3.1}$$

and p_x is defined by

$$\langle x|\,\hat{p} = p_x\,\langle x| \tag{A9.3.2}$$

Therefore

$$\langle x|\,[\hat{x}, \hat{p}] = [x, p_x]\,\langle x| \tag{A9.3.3}$$

and so Eq. (Q9.3.1) becomes

$$[x, p_x] = i \tag{A9.3.4}$$

p_x is a differential operator

$$p_x = \sum_{n=0}^{\infty} A_n(x)\,\partial_x^n \tag{A9.3.5}$$

where $\partial_x \equiv \partial/\partial x$. Substituting into Eq. (A9.3.4) gives $A_n(x) = 0$ for $n > 1$ and $A_1(x) = -i$, and since p_x is Hermitian $A_0(x) = A(x)$ where $A(x)$ is a real function of x. Therefore Eq. (A9.3.4) has the general solution

$$p_x = -i\partial_x + A(x) \tag{A9.3.6}$$

We can express Eq. (Q9.3.3) in terms of p_x using Eq. (A9.3.2)

$$e^{ia\hat{p}}\,|x\rangle = \left(\langle x|\,e^{-ia\hat{p}}\right)^\dagger = \left(e^{-iap_x}\,\langle x|\right)^\dagger = e^{iap_x^*}\,|x\rangle \tag{A9.3.7}$$

and so Eq. (Q9.3.3) essentially reduces to Eq. (Q9.3.4), though note the $*$.

In the special case $A(x) = 0$, Eq. (Q9.3.4) becomes

$$e^{iap_x}\,\phi(x) = e^{a\partial_x}\,\phi(x) = \sum_{n=0}^{\infty} \frac{a^n}{n!}\frac{d^n}{dx^n}\,\phi(x) = \phi(x+a) \tag{A9.3.8}$$

and, using Eq. (A9.3.7), Eq. (Q9.3.3) becomes

$$e^{ia\hat{p}}\,|x\rangle = e^{-a\partial_x}\,|x\rangle = |x-a\rangle \tag{A9.3.9}$$

More generally

$$e^{iap_x} = e^{a[\partial_x + iA(x)]} \tag{A9.3.10}$$

Now

$$\frac{\partial}{\partial a} e^{a[\partial_x + iA(x)]} = e^{a[\partial_x + iA(x)]}\,[\partial_x + i\,A(x)] \tag{A9.3.11}$$

$$= e^{a[\partial_x + iA(x)]}\partial_x + i\,A(x+a)\,e^{a[\partial_x + iA(x)]} \tag{A9.3.12}$$

therefore

$$e^{a[\partial_x + iA(x)]} = e^{i\int_x^{x+a} A(x')dx'}\,e^{a\partial_x} \tag{A9.3.13}$$

Putting everything together, Eq. (Q9.3.4) is evaluated as

$$e^{iap_x}\,\phi(x) = e^{a[\partial_x + iA(x)]}\,\phi(x) = e^{i\int_x^{x+a} A(x')dx'}\,e^{a\partial_x}\,\phi(x) = e^{i\int_x^{x+a} A(x')dx'}\,\phi(x+a) \tag{A9.3.14}$$

and Eq. (Q9.3.3) is evaluated as

$$e^{ia\hat{p}}\,|x\rangle = e^{iap_x^*}\,|x\rangle = e^{-a[\partial_x - iA(x)]}\,|x\rangle = e^{i\int_{x-a}^{x} A(x')dx'}\,e^{-a\partial_x}\,|x\rangle = e^{i\int_{x-a}^{x} A(x')dx'}\,|x-a\rangle \tag{A9.3.15}$$