Homework 9

Answers should be submitted, as both a tex file and a pdf file, to both me and the teaching assistants. You may use this file as a template.

Q9.1. Consider the differential operator

$$L_x = \frac{d^2}{dx^2} \tag{Q9.1.1}$$

acting on functions $\phi : [0, 2\pi] \to \mathbb{C}$.

What is the most general boundary condition consistent with L_x being Hermitian? Show that

i. the Dirichlet boundary conditions

$$\phi(0) = \phi(2\pi) = 0 \tag{Q9.1.2}$$

ii. the Neumann boundary conditions

$$\phi'(0) = \phi'(2\pi) = 0 \tag{Q9.1.3}$$

iii. the periodic boundary conditions

$$\phi(0) = \phi(2\pi) \quad \text{and} \quad \phi'(0) = \phi'(2\pi) \tag{Q9.1.4}$$

are all special cases of this general boundary condition.

In each case

(a) determine the eigenvalues and eigenspaces,

(b) state[1] the orthogonality and completeness equations,

(c) check that $W(\phi, \psi) = \text{constant}$ for ϕ and ψ in the same eigenspace,

(d) express

$$f(x) = 2\pi x - x^2 \tag{Q9.1.5}$$

in terms of eigenvectors of L_x.

Q9.2. The real differential equation

$$\frac{d^2y}{dx^2} + a(x)\,\frac{dy}{dx} + [\lambda\, b(x) + c(x)]\, y = 0 \tag{Q9.2.1}$$

with boundary conditions

$$y(0) = y(1) = 0 \tag{Q9.2.2}$$

has solutions $y_\lambda(x)$. What properties do the $y_\lambda(x)$ have?

[1] It is not necessary to derive.