Homework 9

Q9.1. Consider the differential operator

$$L_x = \frac{d^2}{dx^2} \tag{Q9.1.1}$$

acting on functions $\phi : [0, 2\pi] \to \mathbb{C}$.

What is the most general boundary condition consistent with L_x being Hermitian? Show that

i. the Dirichlet boundary conditions

$$\phi(0) = \phi(2\pi) = 0 \tag{Q9.1.2}$$

ii. the Neumann boundary conditions

$$\phi'(0) = \phi'(2\pi) = 0 \tag{Q9.1.3}$$

iii. the periodic boundary conditions

$$\phi(0) = \phi(2\pi) \quad \text{and} \quad \phi'(0) = \phi'(2\pi) \tag{Q9.1.4}$$

are all special cases of this general boundary condition.

In each case

(a) determine the eigenvalues and eigenspaces,

(b) state the orthogonality and completeness equations,

(c) check that $W(\phi, \psi) =$ constant for ϕ and ψ in the same eigenspace,

(d) express

$$f(x) = 2\pi x - x^2 \tag{Q9.1.5}$$

in terms of eigenvectors of L_x.

A9.1.

$$\phi^* L_x \psi - \psi L_x^* \phi^* = \frac{d}{dx}\left(\phi^* \frac{d\psi}{dx} - \psi \frac{d\phi^*}{dx}\right) \tag{A9.1.1}$$

Therefore, from Eqs. (3.4.8) to (3.4.10), the general Hermitian boundary condition is

$$\phi^*(0)\,\psi'(0) - \psi(0)\,\phi^{*\prime}(0) = \phi^*(2\pi)\,\psi'(2\pi) - \psi(2\pi)\,\phi^{*\prime}(2\pi) \tag{A9.1.2}$$

The Dirichlet, Neumann and periodic boundary conditions, Eqs. (Q9.1.2), (Q9.1.3) and (Q9.1.4), are clearly special cases of this general boundary condition.

(a) The general solution of the eigenvector equation

$$L_x\phi = \lambda\phi \tag{A9.1.3}$$

is

$$\phi_\lambda(x) = \begin{cases} A_0 + B_0 x & \text{for} \quad \lambda = 0 \\ A_\lambda e^{\sqrt{\lambda}x} + B_\lambda e^{-\sqrt{\lambda}x} & \text{for} \quad \lambda \neq 0 \end{cases} \tag{A9.1.4}$$

i. The Dirichlet boundary conditions Eq. (Q9.1.2) constrain the coefficients to

$$A_0 = B_0 = 0 \tag{A9.1.5}$$

and

$$A_\lambda + B_\lambda = 0 \qquad (\lambda \neq 0) \tag{A9.1.6}$$

and the eigenvalues to

$$\lambda = -\frac{n^2}{4} \qquad (n \in \mathbb{Z}, n > 0) \tag{A9.1.7}$$

Therefore the eigenspaces are

$$\phi_\lambda(x) = C_\lambda \sin\left(\frac{nx}{2}\right) \tag{A9.1.8}$$

ii. The Neumann boundary conditions Eq. (Q9.1.3) constrain the coefficients to

$$B_0 = 0 \tag{A9.1.9}$$

and

$$A_\lambda - B_\lambda = 0 \qquad (\lambda \neq 0) \tag{A9.1.10}$$

and the eigenvalues to

$$\lambda = -\frac{n^2}{4} \qquad (n \in \mathbb{Z}, n \geq 0) \tag{A9.1.11}$$

Therefore the eigenspaces are

$$\phi_\lambda(x) = C_\lambda \cos\left(\frac{nx}{2}\right) \tag{A9.1.12}$$

iii. The periodic boundary conditions Eq. (Q9.1.4) constrain the coefficients to

$$B_0 = 0 \tag{A9.1.13}$$

and the eigenvalues to

$$\lambda = -n^2 \qquad (n \in \mathbb{Z}) \tag{A9.1.14}$$

Therefore the eigenspaces are

$$\phi_\lambda(x) = A_\lambda e^{inx} + B_\lambda e^{-inx} \tag{A9.1.15}$$

(b) i. Orthogonality

$$\int_0^{2\pi} dx \sin\left(\frac{mx}{2}\right)\sin\left(\frac{nx}{2}\right) = \pi\delta_{mn} \tag{A9.1.16}$$

Completeness

$$\sum_{n=1}^{\infty} \sin\left(\frac{nx}{2}\right)\sin\left(\frac{ny}{2}\right) = \pi\left[\delta_{4\pi}(x-y) - \delta_{4\pi}(x+y)\right] \tag{A9.1.17}$$

where

$$\delta_p(x) = \delta(x) \qquad \left(-\frac{p}{2} \leq x \leq \frac{p}{2}\right) \tag{A9.1.18}$$

and

$$\delta_p(x) = \delta_p(x+p) \tag{A9.1.19}$$

ii. Orthogonality

$$\int_0^{2\pi} dx \cos\left(\frac{mx}{2}\right)\cos\left(\frac{nx}{2}\right) = \pi\delta_{mn} + \pi\delta_{m0}\delta_{n0} \tag{A9.1.20}$$

Completeness

$$\frac{1}{2} + \sum_{n=1}^{\infty} \cos\left(\frac{nx}{2}\right)\cos\left(\frac{ny}{2}\right) = \pi\left[\delta_{4\pi}(x-y) + \delta_{4\pi}(x+y)\right] \tag{A9.1.21}$$

iii. Orthogonality

$$\int_0^{2\pi} dx\, e^{-imx}e^{inx} = 2\pi\delta_{mn} \tag{A9.1.22}$$

Completeness

$$\sum_{n=-\infty}^{\infty} e^{inx}e^{-iny} = 2\pi\,\delta_{2\pi}(x-y) \tag{A9.1.23}$$

(c)

$$W(\phi,\psi) = \phi^*\psi' - \phi^{*\prime}\psi \tag{A9.1.24}$$

i.

$$W\left(\sin\frac{nx}{2}, \sin\frac{nx}{2}\right) = 0 \tag{A9.1.25}$$

ii.

$$W\left(\cos\frac{nx}{2}, \cos\frac{nx}{2}\right) = 0 \tag{A9.1.26}$$

iii.

$$W\left(e^{inx}, e^{inx}\right) = 2in \tag{A9.1.27}$$

$$W\left(e^{-inx}, e^{inx}\right) = 0 \tag{A9.1.28}$$

(d) i.

$$f(x) = \sum_{n=1}^{\infty} f_n \sin\left(\frac{nx}{2}\right) \tag{A9.1.29}$$

where

$$f_n = \frac{1}{\pi}\int_0^{2\pi} dx \sin\left(\frac{nx}{2}\right) f(x) \tag{A9.1.30}$$

$$= \begin{cases} \dfrac{32}{\pi n^3} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases} \tag{A9.1.31}$$

ii.

$$f(x) = \frac{f_0}{2} + \sum_{n=1}^{\infty} f_n \cos\left(\frac{nx}{2}\right) \tag{A9.1.32}$$

where

$$f_n = \frac{1}{\pi}\int_0^{2\pi} dx \cos\left(\frac{nx}{2}\right) f(x) \tag{A9.1.33}$$

$$= \begin{cases} \dfrac{4\pi^2}{3} & \text{for } n = 0 \\ 0 & \text{for } n \text{ odd} \\ -\dfrac{16}{n^2} & \text{for } n \text{ even, } n > 0 \end{cases} \tag{A9.1.34}$$

iii.

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{inx} \tag{A9.1.35}$$

where

$$f_n = \frac{1}{2\pi}\int_0^{2\pi} dx\, e^{-inx} f(x) \tag{A9.1.36}$$

$$= \begin{cases} \dfrac{2\pi^2}{3} & \text{for } n = 0 \\ -\dfrac{2}{n^2} & \text{for } n \neq 0 \end{cases} \tag{A9.1.37}$$

Q9.2. The real differential equation

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + [\lambda\, b(x) + c(x)]\, y = 0 \tag{Q9.2.1}$$

with boundary conditions

$$y(0) = y(1) = 0 \tag{Q9.2.2}$$

has solutions $y_\lambda(x)$. What properties do the $y_\lambda(x)$ have?

A9.2. Rearranging Eq. (Q9.2.1) gives the eigen equation

$$L_x y = \lambda y \tag{A9.2.1}$$

with

$$L_x = -\frac{1}{b(x)}\frac{d^2}{dx^2} - \frac{a(x)}{b(x)}\frac{d}{dx} - \frac{c(x)}{b(x)} \tag{A9.2.2}$$

Assuming $a(x)$, $b(x)$ and $c(x)$ are real, and using Eqs. (3.4.18) to (3.4.21), L_x has the form of a Hermitian differential operator

$$g(x)\, L_x = -\frac{d}{dx}\, p(x)\, \frac{d}{dx} - q(x) \tag{A9.2.3}$$

with

$$p(x) = e^{\int dx a(x)} \tag{A9.2.4}$$
$$q(x) = c(x)\, e^{\int dx a(x)} \tag{A9.2.5}$$
$$g(x) = b(x)\, e^{\int dx a(x)} \tag{A9.2.6}$$

Therefore the eigenfunctions $y_\lambda(x)$ will be orthogonal [1], over the range determined by the boundary conditions Eq. (Q9.2.2) and with the integration measure of Eq. (A9.2.6),

$$\int_0^1 dx\, b(x)\, e^{\int dx a(x)}\, y_\lambda^*(x)\, y_{\lambda'}(x) \propto \delta_{\lambda\lambda'} \tag{A9.2.7}$$

and complete

$$\sum_\lambda \frac{y_\lambda(x)\, y_\lambda^*(x')}{\int_0^1 dx''\, b(x'')\, e^{\int dx'' a(x'')}\, y_\lambda^*(x'')\, y_\lambda(x'')} = \frac{\delta(x, x')}{b(x)\, e^{\int dx a(x)}} \tag{A9.2.8}$$

[1] Degenerate eigenfunctions can be chosen to be orthogonal, and will also satisfy Eq. (3.4.11). However, the notation $y_\lambda(x)$ would seem to imply that they are not degenerate.