Homework 10

Optional extra questions.

Q10.1. The function ϕ satisfies

$$L\phi = 0 \qquad \text{(Q10.1.1)}$$

and inhomogeneous boundary conditions. The linear operator L has the form

$$L = L_0 - L_1 \qquad \text{(Q10.1.2)}$$

and the function ϕ_{b} satisfying

$$L_0 \phi_{\mathrm{b}} = 0 \qquad \text{(Q10.1.3)}$$

and the inhomogeneous boundary conditions is known. L_0 and L_1 are Hermitian with respect to the homogeneous boundary conditions associated with the inhomogeneous boundary conditions, and the Green's operator G_0 satisfying

$$L_0 G_0 = 1 \qquad \text{(Q10.1.4)}$$

and the homogeneous boundary conditions is known.

(a) Use G_0 to obtain an equation which can be iterated to solve Eq. (Q10.1.1) for small L_1.

(b) Solve the Green's operator equation

$$LG = 1 \qquad \text{(Q10.1.5)}$$

and use G to solve Eq. (Q10.1.1) for small L_1.

Check that your answers are consistent and reexpress your answer in component form in the case that L_0 and L_1 are differential operators.

Q10.2. Consider the differential operator

$$L_x = \frac{d^2}{dx^2} \qquad \text{(Q10.2.1)}$$

acting on functions $\phi : [0, \pi] \to \mathbb{R}$ with boundary condition

$$\frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(\pi) = 0 \qquad \text{(Q10.2.2)}$$

(a) Show that the Green's function equation

$$L_x \, G(x, y) = \delta(x - y) \qquad \text{(Q10.2.3)}$$

has no solution satisfying the boundary condition

$$\frac{\partial G}{\partial x}(0, y) = \frac{\partial G}{\partial x}(\pi, y) = 0 \qquad \text{(Q10.2.4)}$$

Explain why not.

(b) Express the Green's function for L_x in terms of the eigenvectors of L_x.

(c) Hence solve

$$L_x\,\phi(x) = \rho(x) \tag{Q10.2.5}$$

in general, and for

$$\rho(x) = \rho_0 \cos(nx) \tag{Q10.2.6}$$

with $n \in \mathbb{N}$ in particular. What happens if $n = 0$? Explain.

Q10.3. Use the Green's function method to solve

$$\ddot{x} = f(t) \tag{Q10.3.1}$$

with initial conditions

$$x(0) = 0 \tag{Q10.3.2}$$
$$\dot{x}(0) = v \tag{Q10.3.3}$$

Compare your solution with direct integration of Eq. (Q10.3.1). Give a physical interpretation of the Green's function and the Green's function solution.

Hence solve

$$\ddot{x} + \lambda^2(t)\,x = 0 \tag{Q10.3.4}$$

with the same initial conditions, to leading order in $\lambda(t) \ll t^{-1}$. Check your approximate solution using the exact solution in the case $\lambda =$ constant.