Homework 10

Optional extra questions.

Q10.1. The function ϕ satisfies

$$L\phi = 0 \tag{Q10.1.1}$$

and inhomogeneous boundary conditions. The linear operator L has the form

$$L = L_0 - L_1 \tag{Q10.1.2}$$

and the function ϕ_b satisfying

$$L_0\phi_\mathrm{b} = 0 \tag{Q10.1.3}$$

and the inhomogeneous boundary conditions is known. L_0 and L_1 are Hermitian with respect to the homogeneous boundary conditions associated with the inhomogeneous boundary conditions, and the Green's operator G_0 satisfying

$$L_0 G_0 = 1 \tag{Q10.1.4}$$

and the homogeneous boundary conditions is known.

(a) Use G_0 to obtain an equation which can be iterated to solve Eq. (Q10.1.1) for small L_1.

(b) Solve the Green's operator equation

$$LG = 1 \tag{Q10.1.5}$$

and use G to solve Eq. (Q10.1.1) for small L_1.

Check that your answers are consistent and reexpress your answer in component form in the case that L_0 and L_1 are differential operators.

A10.1. Let

$$\phi = \phi_\mathrm{b} + \psi \tag{A10.1.1}$$

so that ψ satisfies the homogeneous boundary conditions which define the Hilbert space in which L_0 is Hermitian.

(a) Since L_0 is Hermitian, Eq. (Q10.1.4) gives

$$G_0^\dagger = G_0^\dagger L_0 G_0 = (L_0 G_0)^\dagger G_0 = G_0 \tag{A10.1.2}$$

therefore

$$G_0 L_0 = (L_0 G_0)^\dagger = 1 \tag{A10.1.3}$$

Using Eq. (Q10.1.2), Eq. (Q10.1.1) becomes

$$L_0\phi = L_1\phi \tag{A10.1.4}$$

and using Eqs. (A10.1.1) and (Q10.1.3), it becomes

$$L_0\psi = L_1\phi \tag{A10.1.5}$$

Inverting using Eq. (A10.1.3) gives

$$\psi = G_0 L_1 \phi \tag{A10.1.6}$$

and so [1]

$$\phi = \phi_\mathrm{b} + G_0 L_1 \phi \tag{A10.1.7}$$

For small L_1, we can iterate to give

$$\phi = \sum_{n=0}^{\infty} (G_0 L_1)^n \phi_\mathrm{b} \tag{A10.1.8}$$

(b) As in Answer A10.1a, since L is Hermitian, Eqs. (Q10.1.5), (Q10.1.2) and (A10.1.3) give

$$G = L^{-1} = (G_0 L)^{-1} G_0 = (1 - G_0 L_1)^{-1} G_0 \tag{A10.1.9}$$

Using Eq. (A10.1.1), Eq. (Q10.1.1) becomes

$$L\psi = -L\phi_\mathrm{b} \tag{A10.1.10}$$

and using Eqs. (Q10.1.2) and (Q10.1.3), it becomes

$$L\psi = L_1\phi_\mathrm{b} \tag{A10.1.11}$$

Inverting using Eq. (A10.1.9) gives

$$\psi = G L_1 \phi_\mathrm{b} \tag{A10.1.12}$$

and so

$$\phi = \phi_\mathrm{b} + G L_1 \phi_\mathrm{b} \tag{A10.1.13}$$

For small L_1, we can expand Eq. (A10.1.9) to give

$$G = \sum_{n=0}^{\infty} (G_0 L_1)^n G_0 \tag{A10.1.14}$$

and so

$$\phi = \phi_\mathrm{b} + \sum_{n=0}^{\infty} (G_0 L_1)^n G_0 L_1 \phi_\mathrm{b} = \sum_{n=0}^{\infty} (G_0 L_1)^n \phi_\mathrm{b} \tag{A10.1.15}$$

in agreement with Eq. (A10.1.8).

[1] Alternatively, we can use Eqs. (Q10.1.3) and (Q10.1.4) to go directly from Eq. (A10.1.4) to Eq. (A10.1.7), but must note that L_0 is neither invertible nor Hermitian on a space containing the functions ϕ and hence one must add the zero mode ϕ_b.

In component form, Eqs. (A10.1.8) and (A10.1.15) become

$$\begin{aligned}\phi(x) &= \phi_{\rm b}(x) + \int dx'\, G_0(x,x')\,(L_1)_{x'}\,\phi_{\rm b}(x') \\ &\quad + \int dx'\, G_0(x,x')\,(L_1)_{x'} \int dx''\, G_0(x',x'')\,(L_1)_{x''}\,\phi_{\rm b}(x'') + \ldots\end{aligned} \tag{A10.1.16}$$

Q10.2. Consider the differential operator

$$L_x = \frac{d^2}{dx^2} \tag{Q10.2.1}$$

acting on functions $\phi : [0,\pi] \to \mathbb{R}$ with boundary condition

$$\frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(\pi) = 0 \tag{Q10.2.2}$$

(a) Show that the Green's function equation

$$L_x\, G(x,y) = \delta(x-y) \tag{Q10.2.3}$$

has no solution satisfying the boundary condition

$$\frac{\partial G}{\partial x}(0,y) = \frac{\partial G}{\partial x}(\pi,y) = 0 \tag{Q10.2.4}$$

Explain why not.

(b) Express the Green's function for L_x in terms of the eigenvectors of L_x.

(c) Hence solve

$$L_x\, \phi(x) = \rho(x) \tag{Q10.2.5}$$

in general, and for

$$\rho(x) = \rho_0 \cos(nx) \tag{Q10.2.6}$$

with $n \in \mathbb{N}$ in particular. What happens if $n = 0$? Explain.

A10.2. (a) The general solution of the homogeneous equation

$$L_x\, \phi_0(x) = 0 \tag{A10.2.1}$$

is

$$\phi_0(x) = A + Bx \tag{A10.2.2}$$

and so the boundary condition Eq. (Q10.2.4) forces

$$G(x,y) \propto \begin{cases} 1 & \text{for} \quad x < y \\ 1 & \text{for} \quad x > y \end{cases} \tag{A10.2.3}$$

therefore it is not possible to obtain the unit discontinuity in $\partial G/\partial x$ at $x = y$ required to satisfy Eq. (Q10.2.3).
Eq. (A10.2.1) with boundary condition Eq. (Q10.2.2) has solution

$$\phi_0(x) = A \tag{A10.2.4}$$

and so L has a zero eigenvalue. However, L is Hermitian and therefore $LG = 1$ implies $G = L^{-1}$ which does not exist.

(b) Using the results of Answer A9.1, the normalised eigenvectors of L_x are

$$\phi_n(x) = \begin{cases} \sqrt{\frac{1}{\pi}} & n = 0 \\ \sqrt{\frac{2}{\pi}} \cos(nx) & n = 1, 2, \ldots \end{cases} \tag{A10.2.5}$$

with eigenvalues $\lambda_n = -n^2$. Therefore

$$G(x, y) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \phi_n(x) \phi_n^*(y) \tag{A10.2.6}$$

$$= -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(nx) \cos(ny) \tag{A10.2.7}$$

(c) Eq. (Q10.2.5) has solution

$$\phi(x) = \phi_0(x) + \int_0^{\pi} dy\, G(x, y)\, \rho(y) \tag{A10.2.8}$$

$$= -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(nx) \int_0^{\pi} dy \cos(ny)\, \rho(y) + \text{constant} \tag{A10.2.9}$$

Therefore in the case of Eq. (Q10.2.6)

$$\phi(x) = -\frac{\rho_0}{n^2} \cos(nx) + \text{constant} \tag{A10.2.10}$$

If $n = 0$ then the right hand side of Eq. (Q10.2.5) is an eigenvector of L with zero eigenvalue. However, L is Hermitian and therefore Eq. (3.6.2) implies Eq. (Q10.2.5) has no solution.

Q10.3. Use the Green's function method to solve

$$\ddot{x} = f(t) \tag{Q10.3.1}$$

with initial conditions

$$x(0) = 0 \tag{Q10.3.2}$$

$$\dot{x}(0) = v \tag{Q10.3.3}$$

Compare your solution with direct integration of Eq. (Q10.3.1). Give a physical interpretation of the Green's function and the Green's function solution.

Hence solve

$$\ddot{x} + \lambda^2(t)\, x = 0 \tag{Q10.3.4}$$

with the same initial conditions, to leading order in $\lambda(t) \ll t^{-1}$. Check your approximate solution using the exact solution in the case $\lambda = \text{constant}$.

A10.3. Set

$$x(t) = x_{\mathrm{i}}(t) + y(t) \tag{A10.3.1}$$

with

$$x_{\mathrm{i}}(t) = vt \tag{A10.3.2}$$

so that $y(t)$ satisfies the homogeneous initial conditions

$$y(0) = \dot{y}(0) = 0 \tag{A10.3.3}$$

necessary for a Hilbert space. The homogeneous equation

$$\ddot{y} = 0 \tag{A10.3.4}$$

has general solution

$$y = A + Bt \tag{A10.3.5}$$

and we can easily construct the Green's function $G(t,t')$ satisfying

$$\frac{\partial^2 G}{\partial t^2}(t,t') = \delta(t-t') \tag{A10.3.6}$$

with initial conditions

$$G(0,t') = \frac{\partial G}{\partial t}(0,t') = 0 \tag{A10.3.7}$$

as

$$G(t,t') = \begin{cases} 0 & \text{for} \quad t < t' \\ t - t' & \text{for} \quad t > t' \end{cases} \tag{A10.3.8}$$

Therefore the Green's function solution is

$$x(t) = x_{\mathrm{i}}(t) + \int_0^\infty dt'\, G(t,t')\, f(t') \tag{A10.3.9}$$

$$= vt + \int_0^t dt'\, (t-t')\, f(t') \tag{A10.3.10}$$

Direct integration of Eq. (Q10.3.1) gives

$$\dot{x}(t) = v + \int_0^t dt'\, f(t') \tag{A10.3.11}$$

$$x(t) = vt + \int_0^t dt' \int_0^{t'} dt''\, f(t'') \tag{A10.3.12}$$

Integrating by parts gives

$$x(t) = vt + \left[t' \int_0^{t'} dt''\, f(t'') \right]_0^t - \int_0^t dt'\, t'\, f(t') \tag{A10.3.13}$$

$$= vt + \int_0^t dt'\, (t - t')\, f(t') \tag{A10.3.14}$$

in agreement with Eq. (A10.3.10).

The Green's function gives the displacement due to a unit impulse applied at time t', while the Green's function solution adds the displacement due to the initial velocity v to the displacements due to the impulses $f(t)\, dt$.

To solve Eq. (Q10.3.4), substitute $f(t) = -\lambda^2(t)\, x(t)$ into Eq. (A10.3.10) to give

$$x(t) = vt - \int_0^t dt'\, (t - t')\, \lambda^2(t')\, x(t') \tag{A10.3.15}$$

For small $\lambda(t)$, to zeroth order

$$x(t) = vt + \mathcal{O}\big(\lambda^2\big) \tag{A10.3.16}$$

Substituting this into the right hand side of Eq. (A10.3.15) gives the solution to leading order in $\lambda(t)$

$$x(t) = vt - v \int_0^t dt'\, t'\, (t - t')\, \lambda^2(t') + \mathcal{O}\big(\lambda^4\big) \tag{A10.3.17}$$

For constant λ, Eq. (A10.3.17) reduces to

$$x(t) = vt - \frac{1}{6}\lambda^2 v t^3 + \mathcal{O}\big(\lambda^4\big) \tag{A10.3.18}$$

in agreement with the exact solution for constant λ

$$x(t) = \frac{v}{\lambda} \sin \lambda t \tag{A10.3.19}$$

$$= vt - \frac{1}{6}\lambda^2 v t^3 + \mathcal{O}\big(\lambda^4\big) \tag{A10.3.20}$$