Chapter 2

Complex Variables

2.1 Holomorphic functions

2.1.1 Complex functions

A complex number z can be written as

$$z = x + iy \tag{2.1.1}$$

where $x, y \in \mathbb{R}$ and $i^2 = -1$. The complex conjugate of z is

$$z^* = x - iy \tag{2.1.2}$$

x and y are the real and imaginary parts of z

$$x = \operatorname{Re} z = \frac{z + z^*}{2} \tag{2.1.3}$$

and

$$y = \operatorname{Im} z = \frac{z - z^*}{2i} \tag{2.1.4}$$

A complex function f of z is a function on the complex plane. Using Eq. (2.1.1), we can think of it as $f(x, y)$, or, using Eqs. (2.1.3) and (2.1.4), as $f(z, z^*)$. The differential of such a function is

$$df = \frac{\partial f}{\partial z}\, dz + \frac{\partial f}{\partial z^*}\, dz^* \tag{2.1.5}$$

where

$$\frac{\partial f}{\partial z} \equiv \lim_{\delta z \to 0} \frac{f(z + \delta z, z^*) - f(z, z^*)}{\delta z} \tag{2.1.6}$$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right) \tag{2.1.7}$$

and

$$\frac{\partial f}{\partial z^*} \equiv \lim_{\delta z^* \to 0} \frac{f(z, z^* + \delta z^*) - f(z, z^*)}{\delta z^*} \tag{2.1.8}$$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right) \tag{2.1.9}$$

2.1.2 Holomorphic functions

A **holomorphic** function is a function purely of z and not z^*

$$\begin{aligned} f &= f(z) && (2.1.10) \\ &= f(x+iy) && (2.1.11) \end{aligned}$$

or equivalently a function satisfying

$$\frac{\partial f}{\partial z^*} = 0 \tag{2.1.12}$$

Note that the limit of dz^*/dz as $dz \to 0$ does not exist, since it takes different values depending on the direction from which zero is approached. Therefore, from Eq. (2.1.5), df/dz exists only if $\partial f/\partial z^* = 0$, in which case $df/dz = \partial f/\partial z$. Thus the mathematical definition of a holomorphic function, that df/dz exists, is equivalent to our definition.

Eqs. (2.1.7) and (2.1.9) give

$$\frac{\partial^2 f}{\partial z \partial z^*} = \frac{1}{4}\nabla^2 f \tag{2.1.13}$$

Therefore both the real and imaginary parts of a holomorphic function are harmonic functions.

2.1.3 Analytic functions

A function is **analytic** at a point z_0 if it can be expanded about z_0 in a Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{2.1.14}$$

with

$$a_n = \frac{1}{n!}\frac{d^n f}{dz^n}(z_0) \tag{2.1.15}$$

A holomorphic function is analytic at any regular point, and its Taylor series converges out to the radius at which it encounters an irregular point. For example, the Taylor expansion

$$\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{2.1.16}$$

converges for all z since $\exp z$ has no irregular points, while

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \tag{2.1.17}$$

diverges for $|z| \geq 1$ since $1/(1-z)$ has a singularity at $z = 1$.

In mathematics, the statement that holomorphic functions are analytic is of fundamental importance. However, from our physical point of view it is trivial, with regular points being defined by this condition. The interest instead lies with what types of irregular points are allowed for a holomorphic function.

2.1.4 Analytic continuation

A complex function $f(z, z^*)$ can be made up of a patchwork of unrelated functions. For example

$$f(z, z^*) = \begin{cases} 0 & \text{for } \operatorname{Re} z < 0 \\ (\operatorname{Re} z)^2 & \text{for } \operatorname{Re} z > 0 \end{cases} \tag{2.1.18}$$

The points on the boundaries connecting the patches, in this case points with $\operatorname{Re} z = 0$, will be irregular since the function cannot be expanded in a Taylor series about those points and a Taylor series cannot be extended across those points.

However, boundaries, or more generally lines, cannot be defined holomorphically. For example, the boundary in Eq. (2.1.18), $\operatorname{Re} z = 0$, is not holomorphic. Thus the irregular points of a holomorphic function can only be isolated points, not lines, and so there can be no obstruction to extending a holomorphic function from a small patch to the whole complex plane. This important property is called **analytic continuation**.

Indeed, one does not even need the function to be defined on a two dimensional patch to analytically continue it to the whole complex plane. For example, if $f(x)$ with $x \in \mathbb{R}$ is known then we can trivially extend it to $f(z)$ with $z \in \mathbb{Z}$. Furthermore, all one really needs to know are the coefficients of the Taylor expansion, so knowing the values of the holomorphic function on an infinite set of points, in a finite domain, is sufficient to uniquely determine it over the whole complex plane.

2.1.5 Singularities

The irregular points of a holomorphic function are isolated and are classified into three types of singularity: poles, essential singularities and branch points.

The simplest type of singularity is a **pole**. For example

$$f(z) = \frac{1}{z^n} \qquad (n \in \mathbb{N}, n > 0) \tag{2.1.19}$$

has a pole at $z = 0$. A holomorphic function can be expanded about a pole in a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{2.1.20}$$

out to the radius at which it encounters another singularity.

A point is an **essential singularity** if and only if the Laurent series about the point has infinitely many negative degree terms. An equivalent definition is that a point $z = z_0$ is an essential singularity if and only if the limit $\lim_{z \to z_0} f(z)$ does not exist nor equals infinity. For example

$$f(z) = e^{1/z} \tag{2.1.21}$$

has an infinite number of negative degree terms in its Laurent expansion about $z = 0$, and $f \to \infty$ as $z \to 0$ along the positive real axis but $f \to 0$ as $z \to 0$ along the negative real axis. Thus f has an essential singularity at $z = 0$.

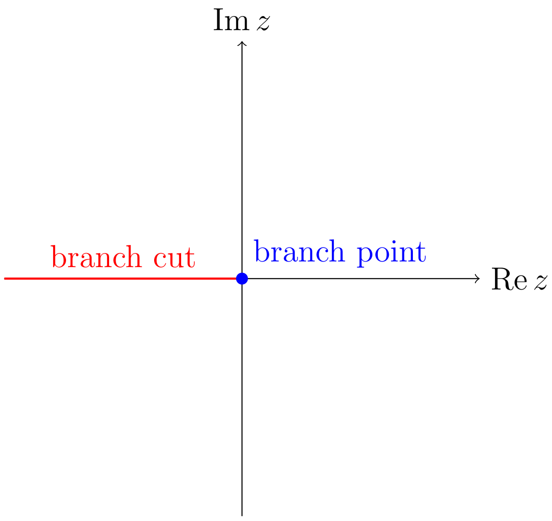

Figure 2.1.1: z^α for $\alpha \notin \mathbb{Z}$ and $\log z$ both have branch points at $z = 0$. The branch cut is usually chosen to lie along the negative real axis.

A function has a **branch point** if the function is not single valued on a loop around the point. For example

$$f(z) = z^\alpha \qquad (\alpha \notin \mathbb{Z}) \tag{2.1.22}$$

acquires a factor $e^{2\pi i\alpha}$, and

$$f(z) = \log z \tag{2.1.23}$$

adds $2\pi i$, as one loops around the origin. To make the function single valued, one can cut the loop with a conveniently chosen branch cut, but then, since the branch cut is not holomorphic, one must not cross the branch cut if one wants to maintain holomorphicity.