2.2 Holomorphic integration

2.2.1 Holomorphic integrals

A holomorphic integral

$$\int_C f(z)\, dz \tag{2.2.1}$$

cannot depend on the curve C since a curve cannot be defined holomorphically. Rather, it depends on the end points, if the curve is open, and possibly on the winding number of the curve around any singularities.

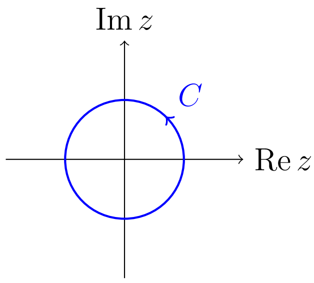

Figure 2.2.1: A loop C with winding number $+1$ around the origin.

The winding number dependence can be seen by explicit integration of a Laurent series term around a closed curve C with winding number N around the origin

$$\int_C z^n\, dz = \begin{cases} \left[\dfrac{z^{n+1}}{n+1}\right]_{\partial C} = 0 & \text{for } n \in \mathbb{Z},\ n \neq -1 \\ \Big[\ln z\Big]_{\partial C} = 2\pi i N & \text{for } n = -1 \end{cases} \tag{2.2.2}$$

Thus only the winding number around simple poles, i.e. those with $n = -1$, is relevant. Note that if $n \notin \mathbb{Z}$ then the branch cut prevents the curve C from being closed and the integral will depend on the end points of the curve.

2.2.2 Contour integration

Holomorphic integration provides a powerful method to evaluate definite integrals, exploiting the fact that the curve can be deformed to a form convenient for evaluation.

Example with a pole

Consider the integral

$$I = \int_0^\infty \frac{dx}{1+x^2} \tag{2.2.3}$$

Deforming the contour as shown in Fig. 2.2.2, we get

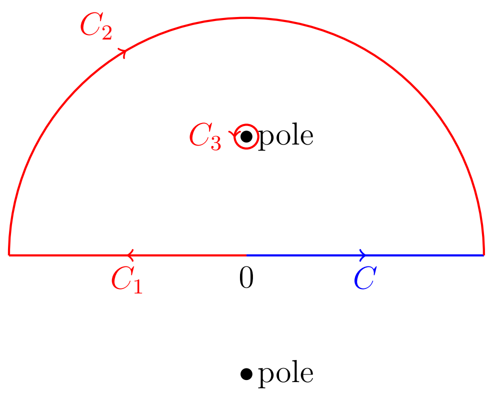

Figure 2.2.2: Contour integration of $\int_0^\infty \frac{dz}{1+z^2}$.

$$\begin{aligned} I &= \int_C \frac{dz}{1+z^2} \\ &= \int_{C_1} \frac{dz}{1+z^2} + \int_{C_2} \frac{dz}{1+z^2} + \int_{C_3} \frac{dz}{1+z^2} \end{aligned} \tag{2.2.4}$$

The last term comes from a loop around the pole, left behind as the contour is deformed across the pole.

By symmetry

$$\int_{C_1} \frac{dz}{1+z^2} = -I \tag{2.2.5}$$

On C_2, the integrand $\sim |z|^{-2}$ and so the integral $\sim |z|^{-1}$. Taking the arc to infinity gives

$$\int_{C_2} \frac{dz}{1+z^2} = 0 \tag{2.2.6}$$

On C_3, $z \sim i$ therefore

$$\frac{1}{1+z^2} \sim \frac{1}{2i}\left(\frac{1}{z-i}\right) \tag{2.2.7}$$

and so from Eq. (2.2.2)

$$\int_{C_3} \frac{dz}{1+z^2} = \frac{1}{2i}(2\pi i)(+1) = \pi \tag{2.2.8}$$

Substituting Eqs. (2.2.5), (2.2.6) and (2.2.8) into Eq. (2.2.4) gives

$$I = -I + 0 + \pi \tag{2.2.9}$$

and so

$$\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2} \tag{2.2.10}$$

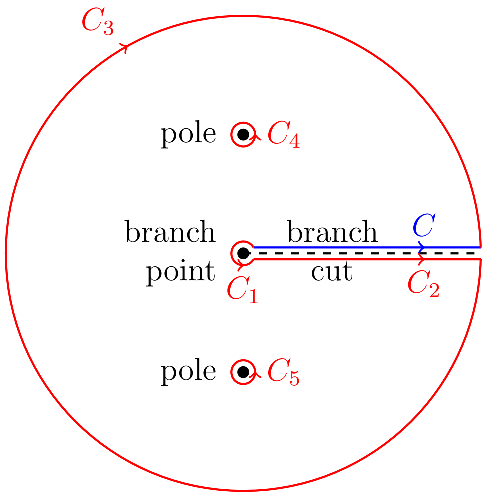

Figure 2.2.3: Contour integration of $\int_0^\infty \frac{z^{\frac{1}{2}}\,dz}{1+z^2}$.

Example with a branch point

Consider the integral

$$I = \int_0^\infty \frac{x^{\frac{1}{2}}\,dx}{1+x^2} \tag{2.2.11}$$

Deforming the contour as shown in Fig. 2.2.3, we get

$$\begin{aligned} I &= \int_C \frac{z^{\frac{1}{2}}\,dz}{1+z^2} \\ &= \int_{C_1} \frac{z^{\frac{1}{2}}\,dz}{1+z^2} + \int_{C_2} \frac{z^{\frac{1}{2}}\,dz}{1+z^2} + \int_{C_3} \frac{z^{\frac{1}{2}}\,dz}{1+z^2} + \int_{C_4} \frac{z^{\frac{1}{2}}\,dz}{1+z^2} + \int_{C_5} \frac{z^{\frac{1}{2}}\,dz}{1+z^2} \end{aligned} \tag{2.2.12}$$

The last two terms come from loops around the poles, left behind as the contour is deformed across the poles.

On C_1, the integrand $\sim |z|^{\frac{1}{2}}$ and so the integral $\sim |z|^{\frac{3}{2}}$. Taking the arc to zero gives

$$\int_{C_1} \frac{z^{\frac{1}{2}}\,dz}{1+z^2} = 0 \tag{2.2.13}$$

C_2 is on the opposite side of the branch cut from C and so

$$z^{\frac{1}{2}}\Big|_{C_2} = -\, z^{\frac{1}{2}}\Big|_C \tag{2.2.14}$$

Therefore

$$\int_{C_2} \frac{z^{\frac{1}{2}}\,dz}{1+z^2} = -I \tag{2.2.15}$$

On C_3, the integrand $\sim |z|^{-\frac{3}{2}}$ and so the integral $\sim |z|^{-\frac{1}{2}}$. Taking the arc to infinity gives

$$\int_{C_3} \frac{z^{\frac{1}{2}}\,dz}{1+z^2} = 0 \tag{2.2.16}$$

On C_4, $z \sim e^{\frac{\pi i}{2}}$ therefore

$$\frac{z^{\frac{1}{2}}}{1+z^2} \sim \frac{e^{\frac{\pi i}{4}}}{2i}\left(\frac{1}{z-i}\right) \tag{2.2.17}$$

and so from Eq. (2.2.2)

$$\int_{C_4} \frac{z^{\frac{1}{2}}\,dz}{1+z^2} = \frac{e^{\frac{\pi i}{4}}}{2i}\,(2\pi i)\,(+1) = \pi e^{\frac{\pi i}{4}} \tag{2.2.18}$$

On C_5, $z \sim e^{\frac{3\pi i}{2}}$ therefore

$$\frac{z^{\frac{1}{2}}}{1+z^2} \sim \frac{e^{\frac{3\pi i}{4}}}{-2i}\left(\frac{1}{z+i}\right) = \frac{e^{-\frac{\pi i}{4}}}{2i}\left(\frac{1}{z+i}\right) \tag{2.2.19}$$

and so from Eq. (2.2.2)

$$\int_{C_5} \frac{z^{\frac{1}{2}}\,dz}{1+z^2} = \frac{e^{-\frac{\pi i}{4}}}{2i}\,(2\pi i)\,(+1) = \pi e^{-\frac{\pi i}{4}} \tag{2.2.20}$$

Substituting Eqs. (2.2.13), (2.2.15), (2.2.16), (2.2.18) and (2.2.20) into Eq. (2.2.12) gives

$$I = 0 - I + 0 + \pi e^{\frac{\pi i}{4}} + \pi e^{-\frac{\pi i}{4}} \tag{2.2.21}$$

and so

$$\int_0^\infty \frac{x^{\frac{1}{2}}\,dx}{1+x^2} = \frac{\pi}{\sqrt{2}} \tag{2.2.22}$$