2.3 Gamma function

2.3.1 Definition and holomorphic structure

The **gamma function** $\Gamma(z)$ is usually represented as the analytic continuation of

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt \qquad (\mathrm{Re}\, z > 0) \tag{2.3.1}$$

Integrating by parts gives

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} \qquad (z \in \mathbb{C}) \tag{2.3.2}$$

Note that although Eq. (2.3.2) was derived from Eq. (2.3.1) under the condition $\mathrm{Re}\, z > 0$, it can be extended to the whole complex plane by analytic continuation. The particular value $\Gamma(1) = 1$ combined with Eq. (2.3.2) gives

$$\Gamma(n+1) = n! \qquad (n \in \mathbb{N}) \tag{2.3.3}$$

The gamma function is the natural generalization of the factorial function to a holomorphic function. The singularities of the gamma function are simple poles at zero and the negative integers, as can be seen using Eqs. (2.3.1) and (2.3.2).

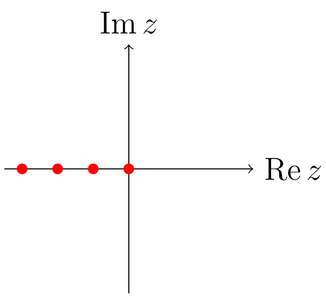

Figure 2.3.1: $\Gamma(z)$ has simple poles at zero and the negative integers.

2.3.2 Saddle point approximation

The asymptotic behavior of the gamma function as $|z| \to \infty$ can be derived from Eq. (2.3.1) using the **saddle point approximation**. Changing the variable of integration to $s = t/z$ gives

$$\Gamma(z) = z^z \int_0^{z^{-1}\infty} s^{-1} e^{z(\ln s - s)}\, ds \qquad (\mathrm{Re}\, z > 0) \tag{2.3.4}$$

which is an integral of the form

$$I(z) = \int_C g(s)\, e^{z f(s)}\, ds \tag{2.3.5}$$

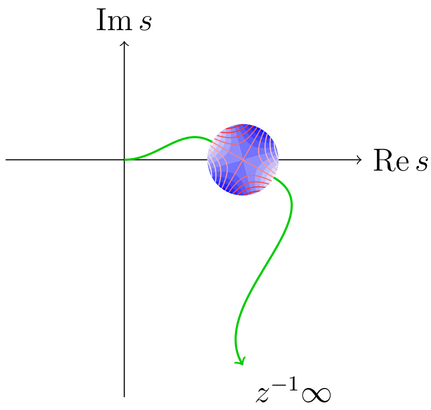

Figure 2.3.2: The curve C is deformed to take the steepest path over the saddle. The integral is then dominated by contributions from the neighborhood of the saddle point. $\mathrm{Re}(zf)$ contours are shaded blue, and $\mathrm{Im}(zf)$ contours are drawn in red, with a stronger color indicating a larger value.

with $C = [0, z^{-1}\infty)$, $g(s) = s^{-1}$ and $f(s) = \ln s - s$. For large $|z|$, the integral will be dominated by contributions from points with maximal $\mathrm{Re}(zf)$, to give the maximal value, and along a path with stationary $\mathrm{Im}(zf)$, to avoid cancellations. Using Eq. (Q2.2.1), the conditions of maximal $\mathrm{Re}(zf)$ and stationary $\mathrm{Im}(zf)$ are satisfied at an extremum[1], $df/ds = 0$, and from Eq. (2.1.13) all extrema are saddle points. Taylor expanding about the relevant saddle point $s = s_0$, we approximate

$$I(z) \sim g(s_0)\, e^{zf(s_0)} \int_C \exp\left[\frac{1}{2} z\, f''(s_0)\, (s - s_0)^2\right] ds \tag{2.3.6}$$

Taking the steepest path over the saddle

$$(s - s_0)^2 = -\frac{u^2}{z\, f''(s_0)} \qquad (u \in \mathbb{R}) \tag{2.3.7}$$

see Figure 2.3.2, and using the Gaussian integral $\int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\, du = \sqrt{2\pi}$, gives

$$I(z) \sim \pm\, g(s_0)\, e^{zf(s_0)} \sqrt{\frac{2\pi}{-z\, f''(s_0)}} \tag{2.3.8}$$

where the sign of the square root is determined by the direction of the path over the saddle.

In the case of the gamma function, Eq. (2.3.4), substituting $s_0 = 1$, $g(s_0) = 1$, $f(s_0) = -1$ and $f''(s_0) = -1$ gives Stirling's formula

$$\Gamma(z) = \sqrt{2\pi}\, e^{-z} z^{z-\frac{1}{2}} \left[1 + \mathcal{O}\left(\frac{1}{z}\right)\right] \qquad (|z| \to \infty, |\arg z| < \pi) \tag{2.3.9}$$

Note that the asymptotic form of the gamma function has a branch cut corresponding to the infinite series of poles of the gamma function along the negative real axis.

[1] Or possibly at an endpoint of C, in which case a slightly different method should be used.

2.3.3 Asymptotic series

Eq. (2.3.9) gives the first term in the **asymptotic series**

$$\ln\Gamma(z) = z\ln z - z - \frac{1}{2}\ln z + \ln\sqrt{2\pi} + \sum_{n=1}^{N}\frac{B_{2n}}{2n(2n-1)z^{2n-1}} + \mathcal{O}\left(\frac{1}{z^{2N+1}}\right) \quad (2.3.10)$$

as $|z| \to \infty$ with $|\arg z| < \pi$. The **Bernoulli numbers**

$$B_{2n} = (-1)^{n+1}\frac{2(2n)!}{(2\pi)^{2n}}\,\zeta(2n) \quad (2.3.11)$$

and the **Riemann zeta function**

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} \qquad (\mathrm{Re}\, s > 1) \quad (2.3.12)$$

The error in Eq. (2.3.10) is $\mathcal{O}\big(z^{-(2N+1)}\big)$ as $|z| \to \infty$ for fixed N, although the series diverges as $N \to \infty$ for fixed $|z|$. For finite $|z|$, there will be an optimum N that minimizes the error, and this optimum N increases as $|z|$ increases. In some sense, although the series is divergent, it converges better for larger $|z|$.

Specifically, for large n the ratio of successive terms is

$$\frac{2n(2n-1)B_{2n+2}}{(2n+2)(2n+1)B_{2n}z^2} \sim -\frac{n^2}{\pi^2 z^2} \quad (2.3.13)$$

and so the optimum N is

$$N \sim \pi|z| \quad (2.3.14)$$