# Chapter 3

# Hilbert spaces

## 3.1 Hilbert spaces

### 3.1.1 Vector spaces

A **vector space** is a set whose elements, called **vectors**, can be added, and multiplied by a scalar, in the usual way. We will focus on complex vector spaces, in which case the scalars are complex numbers.

We will denote a vector $\phi$ by $|\phi\rangle$, and a scalar $\alpha$ by $\alpha$. Then we have the basic operations

$$|\phi\rangle + |\psi\rangle = |\chi\rangle \tag{3.1.1}$$

and

$$\alpha\,|\phi\rangle = |\xi\rangle \tag{3.1.2}$$

### 3.1.2 Hermitian conjugation and covectors

It is natural to extend the concept of complex conjugation to vectors, in which case it is called **Hermitian conjugation** and denoted[1] by a superscript †. The Hermitian conjugate of a vector $|\phi\rangle$ is a **covector** $\langle\phi|$, and vice versa

$$|\phi\rangle^\dagger = \langle\phi| \quad , \quad \langle\phi|^\dagger = |\phi\rangle \tag{3.1.3}$$

The covectors form a dual vector space, with Hermitian conjugation providing an anti-linear bijection between the vectors and the covectors

$$(\alpha\,|\phi\rangle + \beta\,|\psi\rangle)^\dagger = \alpha^*\,\langle\phi| + \beta^*\,\langle\psi| \tag{3.1.4}$$

Vectors and covectors can be **contracted** together to give a scalar

$$\langle\phi|\psi\rangle \in \mathbb{C} \tag{3.1.5}$$

with

$$\langle\chi|\,(\alpha\,|\phi\rangle + \beta\,|\psi\rangle) = \alpha\,\langle\chi|\phi\rangle + \beta\,\langle\chi|\psi\rangle \tag{3.1.6}$$

[1]Mathematicians simply use a superscript $*$.

and

$$\langle\psi|\phi\rangle = (\langle\phi|\psi\rangle)^\dagger = (\langle\phi|\psi\rangle)^* \tag{3.1.7}$$

Combining Hermitian conjugation with contraction gives the **inner product** of two vectors

$$(|\phi\rangle\,, |\psi\rangle) = |\phi\rangle^\dagger\,|\psi\rangle = \langle\phi|\psi\rangle \tag{3.1.8}$$

The magnitude squared of a vector is

$$|\phi|^2 = \langle\phi|\phi\rangle \tag{3.1.9}$$

An inner product on a complex vector space has $|\phi|^2 > 0$ unless $|\phi\rangle = 0$.

A **Hilbert space** is a vector space with an inner product, which is complete, i.e. which contains the limit of any convergent sequence of vectors. Completeness is important because it allows us to use calculus.

### 3.1.3 Subspaces

A **subspace** $\mathcal{G}$ of a Hilbert space $\mathcal{H}$, denoted $\mathcal{G} \subset \mathcal{H}$, is a subset of $\mathcal{H}$ which is also a Hilbert space. The intersection of subspaces, $\mathcal{F} \cap \mathcal{G}$, is also a subspace, but the union, $\mathcal{F} \cup \mathcal{G}$, is not. Instead subspaces can be added

$$\mathcal{F} + \mathcal{G} = \{|f\rangle + |g\rangle : |f\rangle \in \mathcal{F}, |g\rangle \in \mathcal{G}\} \tag{3.1.10}$$

If $\mathcal{H} = \mathcal{F} + \mathcal{G}$ and $\mathcal{F} \cap \mathcal{G} = \{0\}$ then $\mathcal{H}$ is the **direct sum** of $\mathcal{F}$ and $\mathcal{G}$, $\mathcal{H} = \mathcal{F} \oplus \mathcal{G}$, and $\mathcal{F}$ and $\mathcal{G}$ are complementary within $\mathcal{H}$. A complement is not unique but we can use the inner product to define a unique **orthogonal complement** $\mathcal{G}^\perp$ of a subpsace $\mathcal{G} \subset \mathcal{H}$

$$\mathcal{G}^\perp = \{|h\rangle : \langle h|g\rangle = 0 \text{ for all } |g\rangle \in \mathcal{G}\} \tag{3.1.11}$$