3.2 Linear operators

Linear operators are linear mappings from the Hilbert space to itself.

3.2.1 Hermitian conjugate, inverse and commutator

The **Hermitian conjugate** $A^\dagger$ of an operator A is defined by

$$\langle\phi| A^\dagger |\psi\rangle = (\langle\psi| A |\phi\rangle)^* \tag{3.2.1}$$

and the **inverse** A^{-1} of an operator A is defined by

$$A^{-1}A = AA^{-1} = 1 \tag{3.2.2}$$

where 1 is the identity operator.

The **commutator** of two operators A and B is given by

$$[A, B] = AB - BA \tag{3.2.3}$$

They are said to commute if $[A, B] = 0$.

3.2.2 Hermitian, unitary and projection operators

A **Hermitian operator** H has the property

$$H^\dagger = H \tag{3.2.4}$$

and a **unitary operator** U has the property

$$U^\dagger = U^{-1} \tag{3.2.5}$$

A unitary transformation

$$|\phi\rangle \;\to\; U|\phi\rangle \tag{3.2.6}$$
$$\langle\psi| \;\to\; \langle\psi| U^\dagger \tag{3.2.7}$$
$$A \;\to\; UAU^\dagger \tag{3.2.8}$$

leaves contractions invariant

$$\langle\psi| A |\phi\rangle \to \langle\psi| A |\phi\rangle \tag{3.2.9}$$

A **projection operator** P is a Hermitian operator with the property

$$P^2 = P \tag{3.2.10}$$

It projects onto the subspace $\mathcal{P} = \{|\phi\rangle : P|\phi\rangle = |\phi\rangle\}$ and annihilates its orthogonal complement $\mathcal{P}^\perp$

$$P|\phi\rangle = \begin{cases} |\phi\rangle & \text{for} \quad |\phi\rangle \in \mathcal{P} \\ 0 & \text{for} \quad |\phi\rangle \in \mathcal{P}^\perp \end{cases} \tag{3.2.11}$$

For example

$$P = \frac{|\phi\rangle\langle\phi|}{\langle\phi|\phi\rangle} \tag{3.2.12}$$

and $1 - P$ are both projection operators.

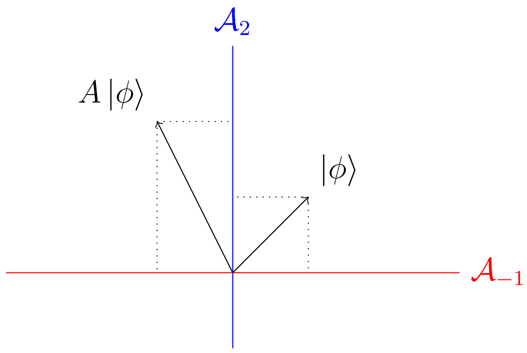

Figure 3.2.1: $A\,|\phi\rangle = -P_{\mathcal{A}_{-1}}\,|\phi\rangle + 2P_{\mathcal{A}_2}\,|\phi\rangle$.

3.2.3 Eigenspaces

An **eigenvector** $|\phi\rangle$ of an operator A satisfies

$$A\,|\phi\rangle = \alpha\,|\phi\rangle \tag{3.2.13}$$

where the **eigenvalue** α is a scalar. Any linear combination of eigenvectors with the same eigenvalue α is also an eigenvector with eigenvalue α, so eigenvectors with the same eigenvalue form a subspace of the Hilbert space called an **eigenspace**

$$\mathcal{A}_\alpha = \{|\phi\rangle : A\,|\phi\rangle = \alpha\,|\phi\rangle\} \tag{3.2.14}$$

The information that Eq. (3.2.13) gives us about A can be distilled as

$$AP_{\mathcal{A}_\alpha} = \alpha P_{\mathcal{A}_\alpha} \tag{3.2.15}$$

where $P_{\mathcal{A}_\alpha}$ is the projection operator that projects onto the eigenspace $\mathcal{A}_\alpha$.

If A is a Hermitian or unitary operator [1] then its eigenspaces $\mathcal{A}_\alpha$ are **orthogonal**

$$P_{\mathcal{A}_\alpha} P_{\mathcal{A}_\beta} = 0 \qquad (\alpha \neq \beta) \tag{3.2.16}$$

and **complete**

$$\sum_\alpha P_{\mathcal{A}_\alpha} = 1 \tag{3.2.17}$$

Therefore, using Eq. (3.2.15), A can be decomposed as

$$A = \sum_\alpha \alpha P_{\mathcal{A}_\alpha} \tag{3.2.18}$$

See Figure 3.2.1.

Let A and B be Hermitian or unitary operators with eigenspaces $\mathcal{A}_\alpha$ and $\mathcal{B}_\beta$ respectively. Then the intersections of their eigenspaces $\mathcal{A}_\alpha \cap \mathcal{B}_\beta$ are orthogonal

$$P_{\mathcal{A}_\alpha \cap \mathcal{B}_\beta} P_{\mathcal{A}_\gamma \cap \mathcal{B}_\delta} = 0 \qquad (\alpha, \beta \neq \gamma, \delta) \tag{3.2.19}$$

[1] Or more generally a normal operator: $[N, N^\dagger] = 0$.

The intersections are complete if and only if A and B commute

$$\sum_{\alpha,\beta} P_{\mathcal{A}_\alpha \cap \mathcal{B}_\beta} = 1 \quad \Leftrightarrow \quad [A, B] = 0 \tag{3.2.20}$$

Thus if A and B commute we can decompose them in terms of a common set of projection operators

$$A = \sum_{\alpha,\beta} \alpha P_{\mathcal{A}_\alpha \cap \mathcal{B}_\beta} \tag{3.2.21}$$

$$B = \sum_{\alpha,\beta} \beta P_{\mathcal{A}_\alpha \cap \mathcal{B}_\beta} \tag{3.2.22}$$

Note that

$$P_{\mathcal{A}_\alpha \cap \mathcal{B}_\beta} = P_{\mathcal{A}_\alpha} P_{\mathcal{B}_\beta} \quad \Leftrightarrow \quad \left[P_{\mathcal{A}_\alpha}, P_{\mathcal{B}_\beta}\right] = 0 \quad \Leftrightarrow \quad [A, B] = 0 \tag{3.2.23}$$