3.3 Bases and components

Hilbert spaces can be finite, countably infinite, or uncountably infinite dimensional. In the first two cases there will be a discrete set of basis vectors, while in the latter case the basis vectors are usually described in terms of continuous parameters.

3.3.1 Discrete bases

A set of **basis vectors** $|\alpha\rangle$ is chosen to be **orthonormal**

$$\langle\alpha|\beta\rangle = \delta_{\alpha\beta} \tag{3.3.1}$$

and **complete**

$$\sum_\alpha |\alpha\rangle\langle\alpha| = 1 \tag{3.3.2}$$

Then an arbitrary vector $|\phi\rangle$ can be expressed in **components** as

$$|\phi\rangle = \sum_\alpha \phi_\alpha |\alpha\rangle \quad \text{where} \quad \phi_\alpha = \langle\alpha|\phi\rangle \tag{3.3.3}$$

Similarly for covectors

$$\langle\phi| = \sum_\alpha \phi_\alpha^* \langle\alpha| \quad \text{where} \quad \phi_\alpha^* = \langle\phi|\alpha\rangle \tag{3.3.4}$$

and linear operators

$$A = \sum_{\alpha,\beta} A_{\alpha\beta} |\alpha\rangle\langle\beta| \quad \text{where} \quad A_{\alpha\beta} = \langle\alpha| A |\beta\rangle \tag{3.3.5}$$

and their Hermitian conjugates[1]

$$A^\dagger = \sum_{\alpha,\beta} A^\dagger_{\alpha\beta} |\alpha\rangle\langle\beta| \quad \text{where} \quad A^\dagger_{\alpha\beta} = A^*_{\beta\alpha} \tag{3.3.6}$$

The contraction of a covector with a vector can be expressed as

$$\langle\phi|\psi\rangle = \sum_\alpha \phi_\alpha^* \psi_\alpha \tag{3.3.7}$$

and a linear operator acting on a vector as

$$A|\phi\rangle = \sum_{\alpha,\beta} A_{\alpha\beta}\phi_\beta |\alpha\rangle \tag{3.3.8}$$

[1] Note that $A^\dagger_{\alpha\beta} \equiv (A^\dagger)_{\alpha\beta} \neq (A_{\alpha\beta})^\dagger = (A_{\alpha\beta})^* \equiv A^*_{\alpha\beta}$.

3.3.2 Continuous bases

When the basis vector label α is a continuous variable, orthonormality becomes

$$\langle\alpha|\beta\rangle = \frac{1}{g(\beta)}\,\delta(\alpha,\beta) \tag{3.3.9}$$

and completeness becomes

$$\int d\alpha\, g(\alpha)\,|\alpha\rangle\,\langle\alpha| = 1 \tag{3.3.10}$$

where $g(\alpha) > 0$ corresponds to the freedom to reparameterize the label α. Usually we take $g(\alpha) = 1$. The delta function $\delta(\alpha,\beta)$ is defined in Section 3.3.3. An arbitrary vector, covector, linear operator or contraction can be expressed in components as

$$|\phi\rangle = \int d\alpha\, g(\alpha)\,\phi(\alpha)\,|\alpha\rangle \quad \text{where} \quad \phi(\alpha) = \langle\alpha|\phi\rangle \tag{3.3.11}$$

$$\langle\phi| = \int d\alpha\, g(\alpha)\,\phi^*(\alpha)\,\langle\alpha| \quad \text{where} \quad \phi^*(\alpha) = \langle\phi|\alpha\rangle \tag{3.3.12}$$

$$A = \int d\alpha\, d\beta\, g(\alpha)\, g(\beta)\, A(\alpha,\beta)\,|\alpha\rangle\,\langle\beta| \quad \text{where} \quad A(\alpha,\beta) = \langle\alpha|\,A\,|\beta\rangle \tag{3.3.13}$$

$$\langle\phi|\psi\rangle = \int d\alpha\, g(\alpha)\,\phi^*(\alpha)\,\psi(\alpha) \tag{3.3.14}$$

3.3.3 Delta function

The **delta function** is defined by

$$\int d\beta\,\delta(\alpha,\beta)\,f(\beta) = f(\alpha) \tag{3.3.15}$$

which is the continuum analogue of $\sum_\beta \delta_{\alpha\beta} f_\beta = f_\alpha$. Its representation depends on the Hilbert space, but is usually given by

$$\delta(\alpha,\beta) = \delta(\beta-\alpha) \tag{3.3.16}$$

or some generalization thereof [2], where

$$\delta(\alpha) = \begin{cases} \infty & \text{if} \quad \alpha = 0 \\ 0 & \text{if} \quad \alpha \neq 0 \end{cases} \tag{3.3.17}$$

with the *simple* divergence at $\alpha = 0$ normalized by

$$\int d\alpha\,\delta(\alpha) = 1 \tag{3.3.18}$$

[2] See Homework 9.

3.3.4 Eigenbases

Due to their orthogonality and completeness, the eigenspaces of a Hermitian or unitary operator, or the intersections of the eigenspaces of a set of commuting Hermitian or unitary operators, provide natural choices of vectors for a basis. In particular, a complete set of commuting Hermitian or unitary operators $A, \ldots, B$, for which all the intersections of the eigenspaces $\mathcal{A}_\alpha \cap \ldots \cap \mathcal{B}_\beta$ are zero or one dimensional, provides a unique [3] orthonormal basis labelled by the eigenvalues of the intersecting eigenspaces

$$|\alpha, \ldots, \beta\rangle \in \mathcal{A}_\alpha \cap \ldots \cap \mathcal{B}_\beta \tag{3.3.19}$$

See Figure 3.3.1.

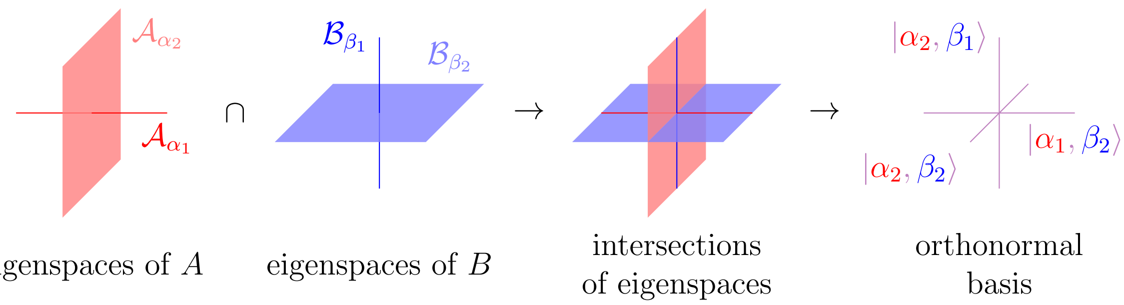

Figure 3.3.1: The intersections of the eigenspaces of a complete set of commuting Hermitian or unitary operators provide a unique orthonormal basis labelled by eigenvalues.

In this basis the operators have the simple diagonalized form

$$\begin{aligned} A &= \sum_{\alpha,\ldots,\beta} \alpha P_{\mathcal{A}_\alpha \cap \ldots \cap \mathcal{B}_\beta} = \sum_{\alpha,\ldots,\beta} \alpha \, |\alpha, \ldots, \beta\rangle \langle \alpha, \ldots, \beta| \\ &\vdots \\ B &= \sum_{\alpha,\ldots,\beta} \beta P_{\mathcal{A}_\alpha \cap \ldots \cap \mathcal{B}_\beta} = \sum_{\alpha,\ldots,\beta} \beta \, |\alpha, \ldots, \beta\rangle \langle \alpha, \ldots, \beta| \end{aligned} \tag{3.3.20}$$

with the trivial action on the basis vectors

$$\begin{aligned} A \, |\alpha, \ldots, \beta\rangle &= \alpha \, |\alpha, \ldots, \beta\rangle \\ &\vdots \\ B \, |\alpha, \ldots, \beta\rangle &= \beta \, |\alpha, \ldots, \beta\rangle \end{aligned} \tag{3.3.21}$$

A simple physical example is given by the operators $\hat{p}_x$, $\hat{p}_y$, $\hat{p}_z$, representing the x, y, z components of a particle's momentum. They form a complete set of commuting Hermitian operators and their mutual eigenvectors $|p_x, p_y, p_z\rangle$, representing a state with momentum (p_x, p_y, p_z), can be used as a basis for the particle's Hilbert space.

[3] Up to signs and ordering.