3.4 Integral and differential operators

3.4.1 Integral operators

An operator L acting on a vector $|\phi\rangle$ is expressed in components as

$$\langle\alpha|\, L\, |\phi\rangle = \int d\beta\, g(\beta)\, L(\alpha,\beta)\, \phi(\beta) \tag{3.4.1}$$

The right hand side of this equation is an **integral operator** $\int d\beta\, g(\beta)\, L(\alpha,\beta)$ acting on a function $\phi(\beta)$. An integral operator is Hermitian if

$$L(\alpha,\beta) = L^*(\beta,\alpha) \tag{3.4.2}$$

3.4.2 Differential operators

Physical operators are often local operators in which case they can be expressed as

$$\langle\alpha|\, L\, |\phi\rangle = L_\alpha\, \phi(\alpha) \tag{3.4.3}$$

where the right hand side is a **differential operator** L_α acting on a function $\phi(\alpha)$. For example

$$\langle x|\, \hat{p}\, |\psi\rangle = -i\hbar \frac{d}{dx}\, \psi(x) \tag{3.4.4}$$

Canceling $|\phi\rangle$ from both sides of Eq. (3.4.3) gives

$$\langle\alpha|\, L = L_\alpha\, \langle\alpha| \tag{3.4.5}$$

Therefore

$$L = \int d\alpha\, g(\alpha)\, |\alpha\rangle\, L_\alpha\, \langle\alpha| \tag{3.4.6}$$

and

$$L(\alpha,\beta) = \frac{1}{g(\beta)} L_\alpha\, \delta(\alpha-\beta) \tag{3.4.7}$$

3.4.3 Hermitian differential operators

Reexpressing Eq. (3.2.1) in component form, the Hermitian conjugate of a differential operator [1] is defined by

$$\int d\alpha\, g(\alpha)\, \phi^*(\alpha)\, L_\alpha^\dagger\, \psi(\alpha) = \int d\alpha\, g(\alpha)\, \psi(\alpha)\, L_\alpha^*\, \phi^*(\alpha) \tag{3.4.8}$$

for all $|\phi\rangle$ and $|\psi\rangle$ in the Hilbert space. Therefore

$$\phi^*(\alpha)\, L_\alpha^\dagger\, \psi(\alpha) - \psi(\alpha)\, L_\alpha^*\, \phi^*(\alpha) = \frac{1}{g(\alpha)} \frac{d}{d\alpha}\, W(\phi,\psi) \tag{3.4.9}$$

[1] Note that $L_\alpha^\dagger \equiv (L^\dagger)_\alpha \neq (L_\alpha)^\dagger = (L_\alpha)^* \equiv L_\alpha^*$ since L_α is a scalar operator.

with

$$[W(\phi,\psi)]_{\text{boundary}} = 0 \tag{3.4.10}$$

Note that if L is Hermitian and $|\phi\rangle$ and $|\psi\rangle$ are in the same eigenspace, then Eq. (3.4.9) implies

$$W(\phi,\psi) = \text{constant} \tag{3.4.11}$$

A second order differential operator has the general form

$$g(\alpha)\, L_\alpha = a(\alpha)\,\frac{d^2}{d\alpha^2} + b(\alpha)\,\frac{d}{d\alpha} + c(\alpha) \tag{3.4.12}$$

Eq. (3.4.9) gives

$$g(\alpha)\, L_\alpha^\dagger = a^*(\alpha)\,\frac{d^2}{d\alpha^2} + \left[2\,a^{*\prime}(\alpha) - b^*(\alpha)\right]\frac{d}{d\alpha} + a^{*\prime\prime}(\alpha) - b^{*\prime}(\alpha) + c^*(\alpha) \tag{3.4.13}$$

with

$$W(\phi,\psi) = a^*(\alpha)\left[\phi^*(\alpha)\,\psi'(\alpha) - \phi^{*\prime}(\alpha)\,\psi(\alpha)\right] + \left[a^{*\prime}(\alpha) - b^*(\alpha)\right]\phi^*(\alpha)\,\psi(\alpha) \tag{3.4.14}$$

satisfying the boundary condition Eq. (3.4.10). If L is Hermitian then

$$a(\alpha) = p(\alpha) \quad , \quad b(\alpha) = p'(\alpha) + i\,r(\alpha) \quad , \quad c(\alpha) = q(\alpha) + \frac{i}{2}\,r'(\alpha) \tag{3.4.15}$$

where $p(\alpha)$, $q(\alpha)$ and $r(\alpha)$ are real functions. Therefore a **Hermitian second order differential operator** has the general form

$$g(\alpha)\, L_\alpha = \frac{d}{d\alpha}\,p(\alpha)\,\frac{d}{d\alpha} + \frac{i}{2}\left[r(\alpha)\,\frac{d}{d\alpha} + \frac{d}{d\alpha}\,r(\alpha)\right] + q(\alpha) \tag{3.4.16}$$

with

$$W(\phi,\psi) = p(\alpha)\left[\phi^*(\alpha)\,\psi'(\alpha) - \phi^{*\prime}(\alpha)\,\psi(\alpha)\right] + i\,r(\alpha)\,\phi^*(\alpha)\,\psi(\alpha) \tag{3.4.17}$$

satisfying the boundary condition Eq. (3.4.10).

Any real second order differential operator

$$L_\alpha = A(\alpha)\,\frac{d^2}{d\alpha^2} + B(\alpha)\,\frac{d}{d\alpha} + C(\alpha) \tag{3.4.18}$$

with $|A(\alpha)| > 0$ can be expressed as a Hermitian operator

$$g(\alpha)\, L_\alpha = \frac{d}{d\alpha}\,p(\alpha)\,\frac{d}{d\alpha} + q(\alpha) \tag{3.4.19}$$

with

$$\frac{p'(\alpha)}{p(\alpha)} = \frac{B(\alpha)}{A(\alpha)} \tag{3.4.20}$$

and

$$g(\alpha) = \frac{p(\alpha)}{A(\alpha)} \quad , \quad q(\alpha) = \frac{C(\alpha)\,p(\alpha)}{A(\alpha)} \tag{3.4.21}$$

if the boundary condition Eq. (3.4.10) is satisfied.

3.4.4 Eigenfunctions

All of our formalism for the eigenvectors of linear operators translates directly to the eigenfunctions of differential operators simply by taking components with respect to a continuous basis $|\alpha\rangle$.

Discrete set of eigenvectors

In the case of a discrete set of eigenvectors $|\psi_a\rangle$, the eigenvector, orthonormality, completeness and eigenvector expansion equations become

$$L\,|\psi_a\rangle = \lambda_a\,|\psi_a\rangle \quad \rightarrow \quad L_\alpha\,\psi_a(\alpha) = \lambda_a\,\psi_a(\alpha) \tag{3.4.22}$$

$$\langle\psi_a|\psi_b\rangle = \delta_{ab} \quad \rightarrow \quad \int d\alpha\, g(\alpha)\,\psi_a^*(\alpha)\,\psi_b(\alpha) = \delta_{ab} \tag{3.4.23}$$

$$\sum_a |\psi_a\rangle\,\langle\psi_a| = 1 \quad \rightarrow \quad \sum_a \psi_a(\alpha)\,\psi_a^*(\beta) = \frac{1}{g(\alpha)}\,\delta(\alpha-\beta) \tag{3.4.24}$$

$$|\phi\rangle = \sum_a \phi_a\,|\psi_a\rangle \quad \rightarrow \quad \phi(\alpha) = \sum_a \phi_a\,\psi_a(\alpha) \tag{3.4.25}$$

$$\phi_a = \langle\psi_a|\phi\rangle \quad \rightarrow \quad \phi_a = \int d\alpha\, g(\alpha)\,\psi_a^*(\alpha)\,\phi(\alpha) \tag{3.4.26}$$

Continuous set of eigenvectors

Similarly, in the case of a continuous set of eigenvectors $|\psi(a)\rangle$, we have

$$L\,|\psi(a)\rangle = \lambda(a)\,|\psi(a)\rangle \quad \rightarrow \quad L_\alpha\,\psi(a,\alpha) = \lambda(a)\,\psi(a,\alpha) \tag{3.4.27}$$

$$\langle\psi(a)|\psi(b)\rangle = \frac{1}{h(a)}\,\delta(a-b) \quad \rightarrow \quad \int d\alpha\, g(\alpha)\,\psi^*(a,\alpha)\,\psi(b,\alpha) = \frac{1}{h(a)}\,\delta(a-b) \tag{3.4.28}$$

$$\int da\, h(a)\,|\psi(a)\rangle\,\langle\psi(a)| = 1 \quad \rightarrow \quad \int da\, h(a)\,\psi(a,\alpha)\,\psi^*(a,\beta) = \frac{1}{g(\alpha)}\,\delta(\alpha-\beta) \tag{3.4.29}$$

$$|\phi\rangle = \int da\, h(a)\,\phi(a)\,|\psi(a)\rangle \quad \rightarrow \quad \phi(\alpha) = \int da\, h(a)\,\phi(a)\,\psi(a,\alpha) \tag{3.4.30}$$

$$\phi(a) = \langle\psi(a)|\phi\rangle \quad \rightarrow \quad \phi(a) = \int d\alpha\, g(\alpha)\,\psi^*(a,\alpha)\,\phi(\alpha) \tag{3.4.31}$$