3.5 Operator equations

Consider the operator equation

$$L\,|\phi\rangle = |f\rangle \tag{3.5.1}$$

Let $\mathcal{L}_\lambda$ and $\mathcal{L}_\lambda^\dagger$ be the eigenspaces of L and $L^\dagger$, and let $P_{\mathcal{L}_\lambda}$ and $P_{\mathcal{L}_\lambda^\dagger}$ be their projection operators. Then $LP_{\mathcal{L}_0} = 0$ and $L^\dagger P_{\mathcal{L}_0^\dagger} = 0$, and hence $P_{\mathcal{L}_0^\dagger} L = 0$. Therefore, applying $P_{\mathcal{L}_0^\dagger}$ to Eq. (3.5.1), we require

$$P_{\mathcal{L}_0^\dagger}\,|f\rangle = 0 \tag{3.5.2}$$

for a solution $|\phi\rangle$ to exist. If a solution exists, we can decompose it as

$$|\phi\rangle = P_{\mathcal{L}_0}\,|\phi\rangle + (1 - P_{\mathcal{L}_0})\,|\phi\rangle \tag{3.5.3}$$

where $P_{\mathcal{L}_0}\,|\phi\rangle$ is undetermined and $(1 - P_{\mathcal{L}_0})\,|\phi\rangle$ is proportional to $|f\rangle$, and hence can be written as $G\,|f\rangle$ for some linear operator G. Thus the formal solution of Eq. (3.5.1) is

$$|\phi\rangle = P_{\mathcal{L}_0}\,|\phi\rangle + G\,|f\rangle \tag{3.5.4}$$

The ambiguity in G coming from Eq. (3.5.2) can be fixed by setting $GP_{\mathcal{L}_0^\dagger} = 0$, in which case applying $P_{\mathcal{L}_0}$ and L to Eq. (3.5.4) give $P_{\mathcal{L}_0}G = 0$ and

$$LG = 1 - P_{\mathcal{L}_0^\dagger} \tag{3.5.5}$$

respectively. We can make this more explicit in two important cases.

3.5.1 Hermitian operator equations

If L is Hermitian then its eigenspaces are orthogonal and complete

$$L = \sum_\lambda \lambda P_{\mathcal{L}_\lambda} \quad \text{and} \quad 1 = \sum_\lambda P_{\mathcal{L}_\lambda} \tag{3.5.6}$$

Eq. (3.5.1) decomposes to

$$\lambda P_{\mathcal{L}_\lambda}\,|\phi\rangle = P_{\mathcal{L}_\lambda}\,|f\rangle \tag{3.5.7}$$

and inverting for $\lambda \neq 0$ and recombining gives Eq. (3.5.4) with

$$G = \sum_{\lambda \neq 0} \lambda^{-1} P_{\mathcal{L}_\lambda} \tag{3.5.8}$$

which is the inverse of L on $\mathcal{L}_0^\perp$.

3.5.2 Green's functions

If

$$\mathcal{L}_0^\dagger = 0 \tag{3.5.9}$$

then Eq. (3.5.5) reduces to

$$LG = 1 \tag{3.5.10}$$

and if L can be expressed as a differential operator then the Green's function $G(\alpha, \beta)$ satisfies

$$L_\alpha \, G(\alpha, \beta) = \frac{1}{g(\alpha)} \, \delta(\alpha - \beta) \tag{3.5.11}$$

and the homogeneous boundary conditions associated with the Hilbert space. Eq. (3.5.11) is easy to solve since the delta function is zero apart from at $\alpha = \beta$

$$L_\alpha \, G(\alpha, \beta) = 0 \qquad \text{for } \alpha < \beta \tag{3.5.12}$$

and

$$L_\alpha \, G(\alpha, \beta) = 0 \qquad \text{for } \alpha > \beta \tag{3.5.13}$$

The Green's function is a solution of Eq. (3.5.12) satisfying the boundary condition at $\alpha < \beta$, patched together with a solution of Eq. (3.5.13) satisfying the boundary condition at $\alpha > \beta$, with the matching conditions at $\alpha = \beta$ fixed by the right hand side of Eq. (3.5.11).

Boundary Green's function

For L_α of the form of Eq. (3.4.12), and given $\zeta(\alpha)$ satisfying [1]

$$L_\alpha \, \zeta(\alpha) = 0 \quad \text{for} \quad \alpha < \beta \tag{3.5.14}$$

including the boundary condition at $\alpha < \beta$, and $\xi(\alpha)$ satisfying

$$L_\alpha \, \xi(\alpha) = 0 \quad \text{for} \quad \alpha > \beta \tag{3.5.15}$$

including the boundary condition at $\alpha > \beta$, the Green's function is

$$G_{\mathrm{b}}(\alpha, \beta) = \begin{cases} \dfrac{\zeta(\alpha) \, \xi(\beta)}{a(\beta) \, [\zeta(\beta) \, \xi'(\beta) - \zeta'(\beta) \, \xi(\beta)]} & \text{for} \quad \alpha < \beta \\[2ex] \dfrac{\zeta(\beta) \, \xi(\alpha)}{a(\beta) \, [\zeta(\beta) \, \xi'(\beta) - \zeta'(\beta) \, \xi(\beta)]} & \text{for} \quad \alpha > \beta \end{cases} \tag{3.5.16}$$

Note that for L_α of the form of Eq. (3.4.19)

$$a(\beta) \, [\zeta(\beta) \, \xi'(\beta) - \zeta'(\beta) \, \xi(\beta)] = W(\zeta^*, \xi) = \text{constant} \tag{3.5.17}$$

[1] Note that Eq. (3.5.14) is possible despite Eq. (3.5.9) if $\zeta(\alpha)$ does not satisfy the boundary condition at $\alpha > \beta$ and so is not in the Hilbert space.

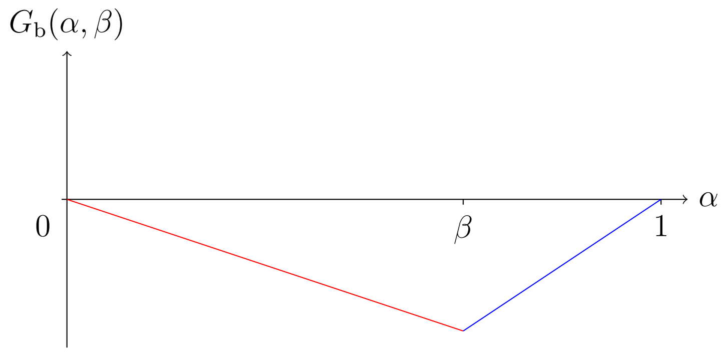

Figure 3.5.1: $\frac{d^2}{d\alpha^2} G_b(\alpha,\beta) = \delta(\alpha-\beta)$ with $G_b(0,\beta) = G_b(1,\beta) = 0$.

For example, for $L_\alpha = d^2/d\alpha^2$ with boundary condition $\phi(0) = \phi(1) = 0$, Eq. (3.5.14) has solution $\zeta(\alpha) \propto \alpha$ and Eq. (3.5.15) has solution $\xi(\alpha) \propto 1-\alpha$ giving the Green's function

$$G_b(\alpha,\beta) = \begin{cases} -(1-\beta)\,\alpha & \text{for} \quad \alpha < \beta \\ -\beta\,(1-\alpha) & \text{for} \quad \alpha > \beta \end{cases} \tag{3.5.18}$$

shown in Figure 3.5.1.

Causal Green's function

For initial condition problems, the solution of Eq. (3.5.14) is zero but Eq. (3.5.15) has two independent solutions, giving the causal Green's function

$$G_c(\alpha,\beta) = \frac{\xi_1(\beta)\,\xi_2(\alpha) - \xi_1(\alpha)\,\xi_2(\beta)}{a(\beta)\,[\xi_1(\beta)\,\xi_2'(\beta) - \xi_1'(\beta)\,\xi_2(\beta)]}\,\theta(\alpha-\beta) \tag{3.5.19}$$

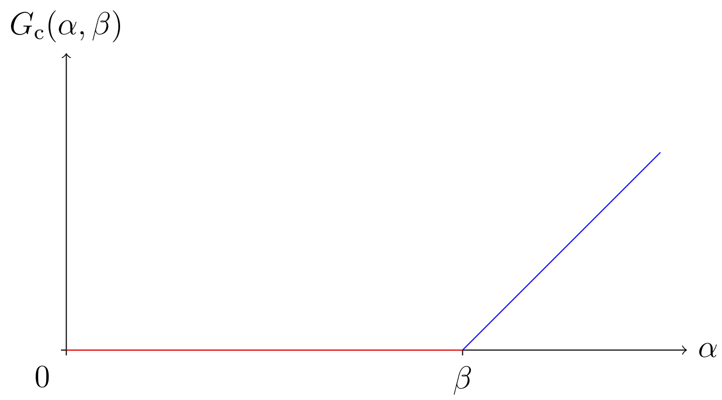

Figure 3.5.2: $\frac{d^2}{d\alpha^2} G_c(\alpha,\beta) = \delta(\alpha-\beta)$ with $G_c(0,\beta) = \frac{dG_c}{d\alpha}(0,\beta) = 0$.

For example, for $L_\alpha = d^2/d\alpha^2$ with initial condition $\phi(0) = \phi'(0) = 0$, Eq. (3.5.15) has solutions $\xi_1(\alpha) \propto 1$ and $\xi_2(\alpha) \propto \alpha$ giving the Green's function

$$G_c(\alpha,\beta) = (\alpha-\beta)\,\theta(\alpha-\beta) \tag{3.5.20}$$

shown in Figure 3.5.2.

3.5.3 Inhomogeneous differential equations

If L can be expressed as a differential operator, then Eq. (3.5.1) takes the component form

$$L_\alpha \, \phi(\alpha) = f(\alpha) \tag{3.5.21}$$

and the Green's function solution Eq. (3.5.4) takes the component form

$$\phi(\alpha) = \phi_0(\alpha) + \int d\beta \, g(\beta) \, G(\alpha, \beta) \, f(\beta) \tag{3.5.22}$$

where $|\phi_0\rangle \in \mathcal{L}_0$.

For example, in the case of Eq. (3.5.18), Eq. (3.5.21) has solution

$$\phi(\alpha) = -\alpha \int_\alpha^1 d\beta \, (1-\beta) \, f(\beta) - (1-\alpha) \int_0^\alpha d\beta \, \beta \, f(\beta) \tag{3.5.23}$$

while in the case of Eq. (3.5.20)

$$\phi(\alpha) = \int_0^\alpha d\beta \, (\alpha - \beta) \, f(\beta) \tag{3.5.24}$$

However, one often needs to solve a differential equation of the form

$$L_\alpha \, \psi(\alpha) = h(\alpha) \tag{3.5.25}$$

where $\psi(\alpha)$ satisfies inhomogeneous boundary conditions, i.e. boundary conditions which are not satisfied by $\psi(\alpha) = 0$, which cannot define a vector space. So first we set

$$\psi(\alpha) = \psi_{\mathrm{b}}(\alpha) + \phi(\alpha) \tag{3.5.26}$$

where $\psi_{\mathrm{b}}(\alpha)$ satisfies the boundary conditions. Then $\phi(\alpha)$ satisfies homogeneous boundary conditions, consistent with a vector space structure. For example, if $\psi(a) = A$ and $\psi(b) = B$, then $\psi_{\mathrm{b}}(a) = A$ and $\psi_{\mathrm{b}}(b) = B$, and $\phi(a) = \phi(b) = 0$. Substituting Eq. (3.5.26) into Eq. (3.5.25) gives Eq. (3.5.21) with

$$f(\alpha) = h(\alpha) - L_\alpha \, \psi_{\mathrm{b}}(\alpha) \tag{3.5.27}$$

and Eq. (3.5.22) gives

$$\psi(\alpha) = \psi_{\mathrm{b}}(\alpha) + \phi_0(\alpha) + \int d\beta \, g(\beta) \, G(\alpha, \beta) \, [h(\beta) - L_\beta \, \psi_{\mathrm{b}}(\beta)] \tag{3.5.28}$$

Note that if L is Hermitian then Eq. (3.5.9) implies $\mathcal{L}_0 = 0$ and hence $\phi_0(\alpha) = 0$, but $\psi_{\mathrm{b}}(\alpha)$ can still be chosen to satisfy $L_\alpha \, \psi_{\mathrm{b}}(\alpha) = 0$ since it is not in the Hilbert space. [2]

[2] Alternatively, we can think in terms of a larger Hilbert space in which L is not Hermitian and choose $\psi_{\mathrm{b}}(\alpha) \in \mathcal{L}_0 \neq \mathcal{L}_0^\dagger$.

3.5.4 Poisson's equation

Poisson's equation

$$-\nabla^2 \phi(\boldsymbol{r}) = \rho(\boldsymbol{r}) \tag{3.5.29}$$

with boundary condition

$$\phi(\boldsymbol{r}) \to 0 \quad \text{as } |\boldsymbol{r}| \to \infty \tag{3.5.30}$$

can be solved using the Green's function $G(\boldsymbol{r}, \boldsymbol{r}')$ satisfying

$$-\nabla^2 G(\boldsymbol{r}, \boldsymbol{r}') = \delta(\boldsymbol{r} - \boldsymbol{r}') \tag{3.5.31}$$

with the boundary condition

$$G(\boldsymbol{r}, \boldsymbol{r}') \to 0 \quad \text{as } |\boldsymbol{r}| \to \infty \tag{3.5.32}$$

Using translational symmetry, we can simplify Eq. (3.5.31) to

$$-\nabla^2 G(\boldsymbol{r}) = \delta(\boldsymbol{r}) \tag{3.5.33}$$

Now

$$\int d^3\boldsymbol{r}\, \delta(\boldsymbol{r}) = \int 4\pi r^2\, dr\, \delta(\boldsymbol{r}) = \int dr\, \delta(r) \tag{3.5.34}$$

and so

$$\delta(\boldsymbol{r}) = \frac{1}{4\pi r^2}\, \delta(r) \tag{3.5.35}$$

Using spherical symmetry and Eq. (3.5.35), Eq. (3.5.33) simplifies to

$$-\frac{d}{dr} r^2 \frac{d}{dr}\, G(r) = \frac{1}{4\pi}\, \delta(r) \tag{3.5.36}$$

The general solution of the homogeneous equation

$$-\frac{d}{dr} r^2 \frac{d}{dr}\, \psi(r) = 0 \tag{3.5.37}$$

is

$$\psi(r) = \frac{A}{r} + B \tag{3.5.38}$$

The boundary condition Eq. (3.5.30) sets $B = 0$ and so

$$G(r) = \frac{A}{r} \qquad \text{for } r > 0 \tag{3.5.39}$$

To fix the normalisation we can integrate Eq. (3.5.36)

$$\begin{aligned} \frac{1}{4\pi} &= \frac{1}{4\pi} \int_0^\infty dr\, \delta(r) && (3.5.40) \\ &= -\int_0^\infty dr\, \frac{d}{dr} r^2 \frac{d}{dr}\, G(r) && (3.5.41) \\ &= \left[-r^2 \frac{d}{dr}\, G(r) \right]_0^\infty && (3.5.42) \\ &= A - 0 && (3.5.43) \end{aligned}$$

since $G(r)$ should be smoothable at $r = 0$. Therefore

$$G(r) = \frac{1}{4\pi r} \qquad (3.5.44)$$

$$G(\boldsymbol{r}) = \frac{1}{4\pi|\boldsymbol{r}|} \qquad (3.5.45)$$

and

$$G(\boldsymbol{r}, \boldsymbol{r}') = \frac{1}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|} \qquad (3.5.46)$$

Therefore the Green's function solution of Eq. (3.5.29) is

$$\phi(\boldsymbol{r}) = \int d^3\boldsymbol{r}'\, G(\boldsymbol{r}, \boldsymbol{r}')\, \rho(\boldsymbol{r}') = \int d^3\boldsymbol{r}'\, \frac{\rho(\boldsymbol{r}')}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|} \qquad (3.5.47)$$

$G(\boldsymbol{r}, \boldsymbol{r}')$ can be interpreted as the electric potential due to a point charge at $\boldsymbol{r} = \boldsymbol{r}'$ and the Green's function solution Eq. (3.5.47) can be interpreted as summing up these contributions over the charge density field $\rho(\boldsymbol{r})$.