3.6 Laplace operator

3.6.1 Hermitian boundary conditions

For the **Laplace operator**

$$\nabla^2 = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \tag{3.6.1}$$

acting on functions $\psi(\boldsymbol{x})$ on a domain V, Eq. (3.4.9) becomes

$$\chi^* \nabla^2 \psi - \psi \nabla^2 \chi^* = \boldsymbol{\nabla} \cdot \boldsymbol{W}(\chi, \psi) \tag{3.6.2}$$

where

$$\boldsymbol{W}(\chi, \psi) = \chi^* \boldsymbol{\nabla} \psi - \psi \boldsymbol{\nabla} \chi^* \tag{3.6.3}$$

Therefore the Laplace operator is Hermitian if

$$\int_{\partial V} \boldsymbol{dS} \cdot \boldsymbol{W} = 0 \tag{3.6.4}$$

where ∂V is the boundary of V. Special cases of this boundary condition include the **Dirichlet** boundary condition

$$[\psi]_{\partial V} = 0 \tag{3.6.5}$$

the **Neumann** boundary condition

$$[\boldsymbol{n} \cdot \boldsymbol{\nabla} \psi]_{\partial V} = 0 \tag{3.6.6}$$

where $\boldsymbol{n}$ is the normal to the boundary, and the **no** or **periodic** boundary condition

$$\partial V = 0 \tag{3.6.7}$$

3.6.2 Eigenfunctions in one dimension

In one dimension,

$$\nabla^2 = \frac{d^2}{dx^2} \tag{3.6.8}$$

If Eq. (3.6.4) is satisfied then ∇^2 is a Hermitian operator and so its eigenspaces

$$\nabla^2 \, \Psi_\lambda(x) = \lambda \, \Psi_\lambda(x) \tag{3.6.9}$$

are orthogonal and complete.

The general solution of Eq. (3.6.9) is

$$\Psi_\lambda(x) = \begin{cases} A_0 + B_0 x & \text{for} \quad \lambda = 0 \\ A_\lambda e^{\sqrt{\lambda}\, x} + B_\lambda e^{-\sqrt{\lambda}\, x} & \text{for} \quad \lambda \neq 0 \end{cases} \tag{3.6.10}$$

Finite domain

For example, take

$$x \in [-R, R] \tag{3.6.11}$$

with the no boundary condition

$$\psi(-R) = \psi(R) \quad , \quad \psi'(-R) = \psi'(R) \tag{3.6.12}$$

so that Eq. (3.6.4) is satisfied. In Eq. (3.6.10), the boundary condition Eq. (3.6.12) constrains $B_0 = 0$ and the eigenvalues to

$$\lambda = -\left(\frac{\pi n}{R}\right)^2 \qquad (n \in \mathbb{Z}) \tag{3.6.13}$$

Therefore the eigenspaces for this finite domain are

$$\Psi_\lambda(x) = A_\lambda \exp\left(\frac{i\pi n x}{R}\right) + B_\lambda \exp\left(-\frac{i\pi n x}{R}\right) \tag{3.6.14}$$

Using

$$\int_{-R}^{R} dx \left[\exp\left(\frac{i\pi n x}{R}\right)\right]^* \exp\left(\frac{i\pi n' x}{R}\right) = 2R\delta_{nn'} \tag{3.6.15}$$

we can choose the set of eigenfunctions [1]

$$\psi_n(x) = \frac{1}{\sqrt{2R}} \exp\left(\frac{i\pi n x}{R}\right) \tag{3.6.16}$$

which are orthonormal

$$\int_{-R}^{R} dx\, \psi_n^*(x)\, \psi_{n'}(x) = \delta_{nn'} \tag{3.6.17}$$

and complete

$$\sum_{n=-\infty}^{\infty} \psi_n(x)\, \psi_n^*(x') = \delta_{2R}(x - x') \tag{3.6.18}$$

where $\delta_{2R}(x - x')$ is the delta function on the domain Eq. (3.6.11) satisfying the boundary conditions Eq. (3.6.12), and we use the notation

$$\delta_p(x) = \delta(x) \qquad \left(-\frac{p}{2} \leq x \leq \frac{p}{2}\right) \tag{3.6.19}$$

and

$$\delta_p(x) = \delta_p(x + p) \tag{3.6.20}$$

A function on the domain Eq. (3.6.11) satisfying the boundary conditions Eq. (3.6.12) can be expanded in terms of the eigenfunctions as

$$f(x) = \sum_{n=-\infty}^{\infty} f_n\, \psi_n(x) \tag{3.6.21}$$

[1] A different choice of eigenfunctions within the eigenspaces could give sines and cosines instead of exponentials.

where

$$f_n = \int_{-R}^{R} dx\, \psi_n^*(x)\, f(x) \tag{3.6.22}$$

which is known as a **Fourier series**.

Infinite domain

Alternatively, taking

$$x \in [-\infty, \infty] \tag{3.6.23}$$

then

$$W(\chi, \psi) = \left(\chi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \chi^*}{\partial x} \right) \tag{3.6.24}$$

is sufficiently well behaved at $x \to \pm\infty$, in the sense that its average value at infinity is zero, if $B_0 = 0$ and

$$\lambda = -k^2 \qquad (k \in \mathbb{R}) \tag{3.6.25}$$

Therefore the eigenspaces for this infinite domain are

$$\Psi_\lambda(x) = A_\lambda e^{ikx} + B_\lambda e^{-ikx} \tag{3.6.26}$$

Using

$$\int_{-\infty}^{\infty} dx \left(e^{ikx}\right)^* e^{ik'x} = 2\pi\, \delta(k - k') \tag{3.6.27}$$

we can choose the set of eigenfunctions

$$\psi(k; x) = \frac{1}{\sqrt{2\pi}} e^{ikx} \tag{3.6.28}$$

which are orthonormal

$$\int_{-\infty}^{\infty} dx\, \psi^*(k; x)\, \psi(k'; x) = \delta(k - k') \tag{3.6.29}$$

and complete

$$\int_{-\infty}^{\infty} dk\, \psi(k; x)\, \psi^*(k; x') = \delta(x - x') \tag{3.6.30}$$

A function on the domain Eq. (3.6.23) which is sufficiently well behaved at $x \to \pm\infty$ can be expanded in terms of the eigenfunctions as

$$f(x) = \int_{-\infty}^{\infty} dk\, \tilde{f}(k)\, \psi(k; x) \tag{3.6.31}$$

where

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\, \psi^*(k; x)\, f(x) \tag{3.6.32}$$

which is known as a **Fourier transform**.

3.6.3 Eigenfunctions in two dimensions

In two dimensional polar coordinates,

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{3.6.33}$$

We identify the operator

$$L_\theta = \frac{\partial^2}{\partial \theta^2} \tag{3.6.34}$$

that commutes with ∇^2

$$\left[\nabla^2, L_\theta\right] = 0 \tag{3.6.35}$$

Eq. (3.4.9) gives

$$W_\theta(\chi, \psi) = \chi^* \frac{\partial \psi}{\partial \theta} - \psi \frac{\partial \chi^*}{\partial \theta} \tag{3.6.36}$$

and L_θ is Hermitian if

$$\left[W_\theta(\chi, \psi)\right]_{\partial C_\theta} = 0 \tag{3.6.37}$$

where ∂C_θ is the boundary of the constant r contours. If Eqs. (3.6.4) and (3.6.37) are satisfied then ∇^2 and L_θ are commuting Hermitian operators and so their eigenspaces

$$\nabla^2 \, \Psi_{\lambda\mu}(r, \theta) = \lambda \, \Psi_{\lambda\mu}(r, \theta) \tag{3.6.38}$$

$$L_\theta \, \Psi_{\lambda\mu}(r, \theta) = \mu \, \Psi_{\lambda\mu}(r, \theta) \tag{3.6.39}$$

are orthogonal and complete.

To solve for the eigenspaces, we note that Eq. (3.6.38) reduces to the ordinary differential equation

$$\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\mu}{r^2}\right) \Psi_{\lambda\mu}(r, \theta) = \lambda \, \Psi_{\lambda\mu}(r, \theta) \tag{3.6.40}$$

Writing $x = \sqrt{-\lambda}\, r$ and $\mu = -n^2$, this takes the form of Bessel's equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + \left(x^2 - n^2\right) y = 0 \tag{3.6.41}$$

whose solutions are the **Bessel functions** [2] of the first $J_n(x)$ and second $Y_n(x)$ kind. Therefore the general solution of Eqs. (3.6.38) and (3.6.39) is

$$\begin{aligned} \Psi_{\lambda\mu}(r, \theta) &= A_{\lambda\mu} \, J_n\!\left(\sqrt{-\lambda}\, r\right) e^{in\theta} + B_{\lambda\mu} \, J_n\!\left(\sqrt{-\lambda}\, r\right) e^{-in\theta} \\ &\quad + C_{\lambda\mu} \, Y_n\!\left(\sqrt{-\lambda}\, r\right) e^{in\theta} + D_{\lambda\mu} \, Y_n\!\left(\sqrt{-\lambda}\, r\right) e^{-in\theta} \end{aligned} \tag{3.6.42}$$

where $n = \sqrt{-\mu}$.

[2] The Bessel functions have many properties which can be looked up as needed.

Finite domain

For example, taking

$$r \in [0, R] \tag{3.6.43}$$

with the boundary condition

$$\psi(R) = 0 \tag{3.6.44}$$

so that Eqs. (3.6.4) and (3.6.37) are satisfied. In Eq. (3.6.42), periodicity in θ fixes

$$n \in \mathbb{Z} \tag{3.6.45}$$

regularity at $r = 0$ fixes $C_{\lambda\mu} = D_{\lambda\mu} = 0$, and the boundary condition Eq. (3.6.44) constrains λ to satisfy

$$J_n\left(\sqrt{-\lambda}\, R\right) = 0 \tag{3.6.46}$$

and so

$$\lambda = -\frac{\zeta_{an}^2}{R^2} \tag{3.6.47}$$

where ζ_{an} is the ath positive zero of $J_n(x)$. Therefore the eigenspaces for this finite domain are

$$\Psi_{\lambda\mu}(r,\theta) = J_n\left(\frac{\zeta_{an} r}{R}\right)\left(A_{\lambda\mu} e^{in\theta} + B_{\lambda\mu} e^{-in\theta}\right) \tag{3.6.48}$$

Using

$$\int_0^1 dx\, x\, \left[J_n(\zeta_{an} x)\right]^2 = \frac{1}{2}\left[J_{n+1}(\zeta_{an})\right]^2 \tag{3.6.49}$$

we can choose the set of eigenvectors

$$\psi_{an}(r,\theta) = \frac{1}{\sqrt{\pi}\, R\, J_{n+1}(\zeta_{an})}\, J_n\left(\frac{\zeta_{an} r}{R}\right) e^{in\theta} \tag{3.6.50}$$

which are orthonormal

$$\int_0^R dr\, r \int_0^{2\pi} d\theta\, \psi_{an}^*(r,\theta)\, \psi_{a'n'}(r,\theta) = \delta_{aa'}\delta_{nn'} \tag{3.6.51}$$

and complete

$$\sum_{a=1}^{\infty} \sum_{n=-\infty}^{\infty} \psi_{an}(r,\theta)\, \psi_{an}^*(r',\theta') = \frac{1}{r}\, \delta(r-r')\, \delta_{2\pi}(\theta-\theta') \tag{3.6.52}$$

A function on the domain Eq. (3.6.98) satisfying the boundary conditions Eq. (3.6.44) can be expanded in terms of the eigenfunctions as

$$f(r,\theta) = \sum_{n=-\infty}^{\infty} \sum_{a=1}^{\infty} f_{an}\, \psi_{an}(r,\theta) \tag{3.6.53}$$

where

$$f_{an} = \int_0^R dr\, r \int_0^{2\pi} d\theta\, \psi_{an}^*(r,\theta)\, f(r,\theta) \tag{3.6.54}$$

which is related to **Fourier-Bessel series**.

Infinite domain

Alternatively, taking
$$r \in [0, \infty) \tag{3.6.55}$$
and noting that
$$J_n(x) \to \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \qquad \text{as } x \to \infty \tag{3.6.56}$$
then
$$W_r(\chi, \psi) = r\left(\chi^* \frac{\partial \psi}{\partial r} - \psi \frac{\partial \chi^*}{\partial r}\right) \tag{3.6.57}$$
is sufficiently well behaved at $r \to \infty$, in the same sense as in Section 3.6.2, if
$$\lambda = -\zeta^2 \qquad (\zeta \in \mathbb{R}) \tag{3.6.58}$$
Therefore the eigenspaces for this infinite domain are
$$\Psi_{\lambda\mu}(r, \theta) = J_n(\zeta r)\left(A_{\lambda\mu} e^{in\theta} + B_{\lambda\mu} e^{-in\theta}\right) \tag{3.6.59}$$

Using
$$\int_0^\infty dr\, r\, J_n(\zeta r)\, J_n(\zeta' r) = \frac{1}{\zeta}\, \delta(\zeta - \zeta') \tag{3.6.60}$$
we can choose the set of eigenvectors
$$\psi_n(\zeta; r, \theta) = \frac{1}{\sqrt{2\pi}}\, J_n(\zeta r)\, e^{in\theta} \tag{3.6.61}$$
which are orthonormal
$$\int_0^\infty dr\, r \int_0^{2\pi} d\theta\, \psi_n^*(\zeta; r, \theta)\, \psi_{n'}(\zeta'; r, \theta) = \frac{1}{\zeta}\, \delta(\zeta - \zeta')\, \delta_{nn'} \tag{3.6.62}$$
and complete
$$\int_0^\infty d\zeta\, \zeta \sum_{n=-\infty}^{\infty} \psi_n(\zeta; r, \theta)\, \psi_n^*(\zeta; r', \theta') = \frac{1}{r}\, \delta(r - r')\, \delta_{2\pi}(\theta - \theta') \tag{3.6.63}$$
A function on the domain Eq. (3.6.55) which is sufficiently well behaved at $r \to \infty$ can be expanded in terms of the eigenfunctions as
$$f(r, \theta) = \int_0^\infty d\zeta\, \zeta \sum_{n=-\infty}^{\infty} f_n(\zeta)\, \psi_n(\zeta; r, \theta) \tag{3.6.64}$$
where
$$f_n(\zeta) = \int_0^\infty dr\, r \int_0^{2\pi} d\theta\, \psi_n^*(\zeta; r, \theta)\, f(r, \theta) \tag{3.6.65}$$
which is related to the **Hankel transform**.

3.6.4 Eigenfunctions in three dimensions

In three dimensional spherical polar coordinates,

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{3.6.66}$$

We identify the operators

$$L_\theta = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{3.6.67}$$

and

$$L_\phi = \frac{\partial^2}{\partial\phi^2} \tag{3.6.68}$$

that commute with ∇^2 and each other

$$\left[\nabla^2, L_\theta\right] = \left[\nabla^2, L_\phi\right] = \left[L_\theta, L_\phi\right] = 0 \tag{3.6.69}$$

Eq. (3.4.9) becomes

$$\chi^* L_\theta\psi - \psi L_\theta\chi^* = \frac{1}{\sin\theta}\boldsymbol{\nabla}_\theta \cdot \boldsymbol{W}_\theta(\chi,\psi) \tag{3.6.70}$$

with

$$\boldsymbol{\nabla}_\theta = \left(\frac{\partial}{\partial\theta}, \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right) \tag{3.6.71}$$

and

$$\boldsymbol{W}_\theta(\chi,\psi) = \sin\theta\left(\chi^*\boldsymbol{\nabla}_\theta\psi - \psi\boldsymbol{\nabla}_\theta\chi^*\right) \tag{3.6.72}$$

and

$$\chi^* L_\phi\psi - \psi L_\phi\chi^* = \frac{\partial}{\partial\phi}W_\phi(\chi,\psi) \tag{3.6.73}$$

with

$$W_\phi(\chi,\psi) = \chi^*\frac{\partial\psi}{\partial\phi} - \psi\frac{\partial\chi^*}{\partial\phi} \tag{3.6.74}$$

Therefore L_θ is Hermitian if

$$\int_{\partial S_\theta} dl\,\boldsymbol{n}\cdot\boldsymbol{W}_\theta = 0 \tag{3.6.75}$$

where ∂S_θ is the boundary of the constant r surfaces, and L_ϕ is Hermitian if

$$\left[W_\phi(\chi,\psi)\right]_{\partial C_\phi} = 0 \tag{3.6.76}$$

where ∂C_ϕ is the boundary of the constant r,θ lines. If Eqs. (3.6.4), (3.6.75) and (3.6.76) are satisfied then ∇^2, L_θ and L_ϕ are commuting Hermitian operators and so their eigenspaces

$$\nabla^2\,\Psi_{\lambda\mu\nu}(r,\theta,\phi) = \lambda\,\Psi_{\lambda\mu\nu}(r,\theta,\phi) \tag{3.6.77}$$

$$L_\theta\,\Psi_{\lambda\mu\nu}(r,\theta,\phi) = \mu\,\Psi_{\lambda\mu\nu}(r,\theta,\phi) \tag{3.6.78}$$

$$L_\phi\,\Psi_{\lambda\mu\nu}(r,\theta,\phi) = \nu\,\Psi_{\lambda\mu\nu}(r,\theta,\phi) \tag{3.6.79}$$

are orthogonal and complete.

To solve for the eigenspaces, we note that Eq. (3.6.77) reduces to the ordinary differential equation

$$\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{\mu}{r^2}\right)\Psi_{\lambda\mu\nu}(r,\theta,\phi)=\lambda\,\Psi_{\lambda\mu\nu}(r,\theta,\phi) \tag{3.6.80}$$

Writing $x=\sqrt{-\lambda}\,r$ and $\mu=-l(l+1)$, this takes the form of the spherical Bessel equation

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+\left[x^2-l(l+1)\right]y=0 \tag{3.6.81}$$

whose solutions are the **spherical Bessel functions** of the first $j_l(x)$ and second $y_l(x)$ kind. Also, Eq. (3.6.78) reduces to the ordinary differential equation

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{\nu}{\sin^2\theta}\right)\Psi_{\lambda\mu\nu}(r,\theta,\phi)=\mu\,\Psi_{\lambda\mu\nu}(r,\theta,\phi) \tag{3.6.82}$$

Writing $x=\cos\theta$, $\mu=-l(l+1)$ and $\nu=-m^2$, this takes the form of the general Legendre equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+\left[l(l+1)-\frac{m^2}{1-x^2}\right]y=0 \tag{3.6.83}$$

whose solutions are the **associated Legendre polynomials** of the first $P_l^m(x)$ and second $Q_l^m(x)$ kind. Therefore the general solution of Eqs. (3.6.77), (3.6.78) and (3.6.79) is

$$\begin{aligned}\Psi_{\lambda\mu\nu}(r,\theta,\phi) \;=\;& A_{\lambda\mu\nu}\,j_l\!\left(\sqrt{-\lambda}\,r\right)P_l^m(\cos\theta)\,e^{im\phi}+B_{\lambda\mu\nu}\,j_l\!\left(\sqrt{-\lambda}\,r\right)P_l^m(\cos\theta)\,e^{-im\phi}\\ &+C_{\lambda\mu\nu}\,j_l\!\left(\sqrt{-\lambda}\,r\right)Q_l^m(\cos\theta)\,e^{im\phi}+D_{\lambda\mu\nu}\,j_l\!\left(\sqrt{-\lambda}\,r\right)Q_l^m(\cos\theta)\,e^{-im\phi}\\ &+E_{\lambda\mu\nu}\,y_l\!\left(\sqrt{-\lambda}\,r\right)P_l^m(\cos\theta)\,e^{im\phi}+F_{\lambda\mu\nu}\,y_l\!\left(\sqrt{-\lambda}\,r\right)P_l^m(\cos\theta)\,e^{-im\phi}\\ &+G_{\lambda\mu\nu}\,y_l\!\left(\sqrt{-\lambda}\,r\right)Q_l^m(\cos\theta)\,e^{im\phi}+H_{\lambda\mu\nu}\,y_l\!\left(\sqrt{-\lambda}\,r\right)Q_l^m(\cos\theta)\,e^{-im\phi}\end{aligned} \tag{3.6.84}$$

where $l=(\sqrt{1-4\mu}-1)/2$ and $m=\sqrt{-\nu^2}$.

Finite domain

For example, taking

$$r\in[0,R] \tag{3.6.85}$$

with the boundary condition

$$\psi(R)=0 \tag{3.6.86}$$

so that Eqs. (3.6.4), (3.6.75) and (3.6.76) are satisfied. In Eq. (3.6.84), periodicity in ϕ fixes

$$m\in\mathbb{Z} \tag{3.6.87}$$

regularity at $\theta = 0, \pi$ fixes $C_{\lambda\mu\nu} = D_{\lambda\mu\nu} = G_{\lambda\mu\nu} = H_{\lambda\mu\nu} = 0$, and

$$l \in \mathbb{Z} \quad \text{and} \quad |m| \leq l \tag{3.6.88}$$

regularity at $r = 0$ fixes $E_{\lambda\mu\nu} = F_{\lambda\mu\nu} = G_{\lambda\mu\nu} = H_{\lambda\mu\nu} = 0$, and the boundary condition Eq. (3.6.86) constrains λ to satisfy

$$j_l\left(\sqrt{-\lambda}\, R\right) = 0 \tag{3.6.89}$$

and so

$$\lambda = -\frac{\zeta_{al}^2}{R^2} \tag{3.6.90}$$

where ζ_{al} is the ath positive zero of $j_l(x)$. Therefore the eigenspaces for this finite domain are

$$\Psi_{\lambda\mu\nu}(r,\theta,\phi) = j_l\left(\frac{\zeta_{al} r}{R}\right) P_l^m(\cos\theta) \left(A_{\lambda\mu\nu} e^{im\phi} + B_{\lambda\mu\nu} e^{-im\phi}\right) \tag{3.6.91}$$

Using

$$\int_0^1 dx\, x^2 \left[j_l(\zeta_{al} x)\right]^2 = \frac{1}{2} \left[j_{l+1}(\zeta_{al})\right]^2 \tag{3.6.92}$$

and

$$\int_{-1}^1 dx \left[P_l^m(x)\right]^2 = \frac{2(l+m)!}{(2l+1)(l-m)!} \tag{3.6.93}$$

we can choose the set of eigenvectors

$$\psi_{alm}(r,\theta,\phi) = \sqrt{\frac{2}{R^3}}\, \frac{1}{j_{l+1}(\zeta_{al})}\, j_l\left(\frac{\zeta_{al} r}{R}\right) Y_l^m(\theta,\phi) \tag{3.6.94}$$

where the $Y_l^m(\theta,\phi)$ are the **spherical harmonics**

$$Y_l^m(\theta,\phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}\; P_l^m(\cos\theta)\, e^{im\phi} \tag{3.6.95}$$

These eigenvectors are orthonormal

$$\int_0^R dr\, r^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi\, \psi_{alm}^*(r,\theta,\phi)\, \psi_{a'l'm'}(r,\theta,\phi) = \delta_{aa'}\delta_{ll'}\delta_{mm'} \tag{3.6.96}$$

and complete

$$\sum_{a=1}^\infty \sum_{l=0}^\infty \sum_{m=-l}^l \psi_{alm}(r,\theta,\phi)\, \psi_{alm}^*(r',\theta',\phi') = \frac{1}{r^2\sin\theta}\, \delta(r-r')\, \delta(\theta-\theta')\, \delta(\phi-\phi') \tag{3.6.97}$$

Infinite domain

Alternatively, taking

$$r \in [0, \infty) \tag{3.6.98}$$

and noting that

$$j_l(x) \to \frac{1}{x} \sin\left(x - \frac{l\pi}{2}\right) \qquad \text{as } x \to \infty \tag{3.6.99}$$

then

$$W_r(\chi, \psi) = r^2 \left(\chi^* \frac{\partial \psi}{\partial r} - \psi \frac{\partial \chi^*}{\partial r}\right) \tag{3.6.100}$$

is sufficiently well behaved at $r \to \infty$, in the same sense as in Section 3.6.2, if

$$\lambda = -\zeta^2 \qquad (\zeta \in \mathbb{R}) \tag{3.6.101}$$

Therefore the eigenspaces for this infinite domain are

$$\Psi_{\lambda\mu\nu}(r, \theta, \phi) = j_l(\zeta r)\, P_l^m(\cos\theta) \left(A_{\lambda\mu\nu} e^{im\phi} + B_{\lambda\mu\nu} e^{-im\phi}\right) \tag{3.6.102}$$

Using

$$\int_0^\infty dx\, x^2\, j_l(\zeta x)\, j_l(\zeta' x) = \frac{\pi}{2\zeta^2}\, \delta(\zeta - \zeta') \tag{3.6.103}$$

we can choose the set of eigenvectors

$$\psi_{lm}(\zeta; r, \theta) = \sqrt{\frac{2}{\pi}}\, j_l(\zeta r)\, Y_l^m(\theta, \phi) \tag{3.6.104}$$

which are orthonormal

$$\int_0^\infty dr\, r^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi\, \psi_{lm}^*(\zeta; r, \theta, \phi)\, \psi_{l'm'}(\zeta'; r, \theta, \phi) = \frac{1}{\zeta^2}\, \delta(\zeta - \zeta')\, \delta_{ll'} \delta_{mm'} \tag{3.6.105}$$

and complete

$$\int_0^\infty d\zeta\, \zeta^2 \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \psi_n(\zeta; r, \theta, \phi)\, \psi_n^*(\zeta; r', \theta', \phi') = \frac{1}{r^2 \sin\theta}\, \delta(r - r')\, \delta(\theta - \theta')\, \delta(\phi - \phi') \tag{3.6.106}$$