Final Exam - 1pm Sunday 15th December, Creative 311

Your answers should be clear and concise. They should start from basic principles and proceed logically. You may cite material from the lecture notes and homework answers.

Q1. Explain the topological and physical meaning of

$$\underline{\nabla} \wedge \underline{\underline{\underline{J}}} = 0 \tag{Q1.1}$$

Q2. Evaluate $g^{\alpha}_{\mathbf{a}}$.

A2.

$$g^{\alpha}_{\mathbf{a}} = g_{\mathbf{ab}} g^{\alpha\mathbf{b}} = \delta^{\alpha}_{\mathbf{a}} = \delta^{\alpha}_{\beta} e^{\beta}_{\mathbf{a}} = e^{\alpha}_{\mathbf{a}} \tag{A2.1}$$

Q3. Show that Eqs. (2.1.16) to (2.1.18) imply Eq. (1.2.4) for $\boldsymbol{\omega}$

(a) a scalar

(b) a covector

A3. (a) Using Eqs. (2.1.16), (2.1.17) and (2.2.3),

$$\left(\underline{\nabla} \wedge \underline{\nabla} \wedge \omega\right)_{[\mathbf{ab}]} = \nabla_{\mathbf{a}} \nabla_{\mathbf{b}} \omega - \nabla_{\mathbf{b}} \nabla_{\mathbf{a}} \omega = 0 \tag{A3.1}$$

(b) Using Eqs. (2.1.17), (2.1.18) and (Q10.2.1),

$$\begin{aligned} \left(\underline{\nabla} \wedge \underline{\nabla} \wedge \underline{\omega}\right)_{[\mathbf{abc}]} &= \nabla_{\mathbf{a}} \left(\nabla_{\mathbf{b}} \omega_{\mathbf{c}} - \nabla_{\mathbf{c}} \omega_{\mathbf{b}}\right) + \nabla_{\mathbf{b}} \left(\nabla_{\mathbf{c}} \omega_{\mathbf{a}} - \nabla_{\mathbf{a}} \omega_{\mathbf{c}}\right) \\ &\quad + \nabla_{\mathbf{c}} \left(\nabla_{\mathbf{a}} \omega_{\mathbf{b}} - \nabla_{\mathbf{b}} \omega_{\mathbf{a}}\right) && \text{(A3.2)} \\ &= R_{\mathbf{abc}}{}^{\mathbf{d}} \omega_{\mathbf{d}} + R_{\mathbf{bca}}{}^{\mathbf{d}} \omega_{\mathbf{d}} + R_{\mathbf{cab}}{}^{\mathbf{d}} \omega_{\mathbf{d}} \;=\; 0 && \text{(A3.3)} \end{aligned}$$

Q4. Express $\triangle \underline{\underline{\omega}}$ in terms of $\nabla^2 \underline{\underline{\omega}}$.

A4. Using Eqs. (2.2.3) and (2.2.4),

$$\begin{aligned} \left(\nabla_{\mathbf{a}} \nabla_{\mathbf{b}} - \nabla_{\mathbf{b}} \nabla_{\mathbf{a}}\right) \omega_{\mathbf{cd}} &= \left(\nabla_{\mathbf{a}} \nabla_{\mathbf{b}} - \nabla_{\mathbf{b}} \nabla_{\mathbf{a}}\right) \left(\omega_{\gamma\delta} e^{\gamma}_{\mathbf{c}} e^{\delta}_{\mathbf{d}}\right) && \text{(A4.1)} \\ &= \omega_{\gamma\delta} e^{\delta}_{\mathbf{d}} \left(\nabla_{\mathbf{a}} \nabla_{\mathbf{b}} - \nabla_{\mathbf{b}} \nabla_{\mathbf{a}}\right) e^{\gamma}_{\mathbf{c}} + \omega_{\gamma\delta} e^{\gamma}_{\mathbf{c}} \left(\nabla_{\mathbf{a}} \nabla_{\mathbf{b}} - \nabla_{\mathbf{b}} \nabla_{\mathbf{a}}\right) e^{\delta}_{\mathbf{d}} && \text{(A4.2)} \\ &= \omega_{\gamma\delta} e^{\delta}_{\mathbf{d}} R_{\mathbf{abc}}{}^{\mathbf{e}} e^{\gamma}_{\mathbf{e}} + \omega_{\gamma\delta} e^{\gamma}_{\mathbf{c}} R_{\mathbf{abd}}{}^{\mathbf{e}} e^{\delta}_{\mathbf{e}} && \text{(A4.3)} \\ &= R_{\mathbf{abc}}{}^{\mathbf{e}} \omega_{\mathbf{ed}} + R_{\mathbf{abd}}{}^{\mathbf{e}} \omega_{\mathbf{ce}} && \text{(A4.4)} \end{aligned}$$

therefore, using Eqs. (Q10.3.1) and (2.1.57), (2.1.12), (2.1.17) and (2.1.18), (2.2.2), (A10.1.8) and (Q10.2.1),

$$\begin{aligned}
\triangle\omega_{[\mathbf{ab}]} &= -\left[\diamond\underline{\nabla}\cdot\diamond\left(\underline{\nabla}\wedge\underline{\underline{\omega}}\right)\right]_{[\mathbf{ab}]} - \left[\underline{\nabla}\wedge\left(\diamond\underline{\nabla}\cdot\diamond\underline{\underline{\omega}}\right)\right]_{[\mathbf{ab}]} && \text{(A4.5)}\\
&= -g_{\mathbf{ag}}g_{\mathbf{bh}}\nabla_{\mathbf{f}}\left[g^{\mathbf{fc}}g^{\mathbf{gd}}g^{\mathbf{he}}\left(\nabla_{\mathbf{c}}\omega_{[\mathbf{de}]}+\nabla_{\mathbf{d}}\omega_{[\mathbf{ec}]}+\nabla_{\mathbf{e}}\omega_{[\mathbf{cd}]}\right)\right]\\
&\quad -\left\{\nabla_{\mathbf{a}}\left[g_{\mathbf{bf}}\nabla_{\mathbf{e}}\left(g^{\mathbf{ec}}g^{\mathbf{fd}}\omega_{[\mathbf{cd}]}\right)\right]-\nabla_{\mathbf{b}}\left[g_{\mathbf{af}}\nabla_{\mathbf{e}}\left(g^{\mathbf{ec}}g^{\mathbf{fd}}\omega_{[\mathbf{cd}]}\right)\right]\right\} && \text{(A4.6)}\\
&= -g^{\mathbf{fc}}\nabla_{\mathbf{f}}\nabla_{\mathbf{c}}\omega_{[\mathbf{ab}]}-g^{\mathbf{fc}}\nabla_{\mathbf{f}}\nabla_{\mathbf{a}}\omega_{[\mathbf{bc}]}-g^{\mathbf{fc}}\nabla_{\mathbf{f}}\nabla_{\mathbf{b}}\omega_{[\mathbf{ca}]}\\
&\quad -g^{\mathbf{ec}}\nabla_{\mathbf{a}}\nabla_{\mathbf{e}}\omega_{[\mathbf{cb}]}+g^{\mathbf{ec}}\nabla_{\mathbf{b}}\nabla_{\mathbf{e}}\omega_{[\mathbf{ca}]} && \text{(A4.7)}\\
&= -\nabla^2\omega_{[\mathbf{ab}]}+g^{\mathbf{ec}}\left(\nabla_{\mathbf{e}}\nabla_{\mathbf{a}}-\nabla_{\mathbf{a}}\nabla_{\mathbf{e}}\right)\omega_{[\mathbf{cb}]}+g^{\mathbf{ec}}\left(\nabla_{\mathbf{b}}\nabla_{\mathbf{e}}-\nabla_{\mathbf{e}}\nabla_{\mathbf{b}}\right)\omega_{[\mathbf{ca}]} && \text{(A4.8)}\\
&= -\nabla^2\omega_{[\mathbf{ab}]}+g^{\mathbf{ec}}R_{\mathbf{eac}}{}^{\mathbf{d}}\omega_{[\mathbf{db}]}+g^{\mathbf{ec}}R_{\mathbf{eab}}{}^{\mathbf{d}}\omega_{[\mathbf{cd}]}+g^{\mathbf{ec}}R_{\mathbf{bec}}{}^{\mathbf{d}}\omega_{[\mathbf{da}]}+g^{\mathbf{ec}}R_{\mathbf{bea}}{}^{\mathbf{d}}\omega_{[\mathbf{cd}]}\\
& && \text{(A4.9)}\\
&= -\nabla^2\omega_{[\mathbf{ab}]}+R_{\mathbf{a}}{}^{\mathbf{d}}\omega_{[\mathbf{db}]}+R_{\mathbf{b}}{}^{\mathbf{d}}\omega_{[\mathbf{ad}]}-R_{\mathbf{ab}}{}^{\mathbf{cd}}\omega_{[\mathbf{cd}]} && \text{(A4.10)}
\end{aligned}$$

Q5. Show that

$$\triangle\vec{v} = -\vec{e}_\alpha g^{\alpha\beta}\frac{\partial}{\partial x^\beta}\left[\frac{1}{\epsilon_{1\cdots N}}\frac{\partial}{\partial x^\gamma}\left(\epsilon_{1\cdots N}v^\gamma\right)\right]-\frac{\vec{e}_\beta}{\epsilon_{1\cdots N}}\frac{\partial}{\partial x^\alpha}\left[\epsilon_{1\cdots N}g^{[\alpha\beta][\gamma\delta]}\frac{\partial}{\partial x^\gamma}\left(g_{\delta\epsilon}v^\epsilon\right)\right] \tag{Q5.1}$$

Hence derive the textbook formula for $\nabla^2\vec{v}$ in two dimensional polar coordinates.

A5. Generalising Eqs. (Q10.3.1),

$$\triangle\vec{v} = -\diamond\underline{\nabla}\wedge\diamond\left(\underline{\nabla}\cdot\vec{v}\right)-\underline{\nabla}\cdot\left(\diamond\underline{\nabla}\wedge\diamond\vec{v}\right) \tag{A5.1}$$

Eq. (Q6.2.1) is

$$\underline{\nabla}\cdot\vec{v} = \frac{1}{\epsilon_{1\cdots N}}\frac{\partial}{\partial x^\alpha}\left(\epsilon_{1\cdots N}v^\alpha\right) \tag{A5.2}$$

therefore

$$\diamond\underline{\nabla}\wedge\diamond\left(\underline{\nabla}\cdot\vec{v}\right) = \vec{e}_\alpha g^{\alpha\beta}\frac{\partial}{\partial x^\beta}\left[\frac{1}{\epsilon_{1\cdots N}}\frac{\partial}{\partial x^\gamma}\left(\epsilon_{1\cdots N}v^\gamma\right)\right] \tag{A5.3}$$

Eqs. (1.4.7), (1.4.4) and (1.4.5), (1.5.8) and (1.5.13), (1.5.22), and (Q5.3.2) give

$$\begin{aligned}
\underline{\nabla} \cdot \vec{\vec{v}} &= -\star^{-1} \underline{\nabla} \wedge \star\vec{\vec{v}} && \text{(A5.4)}\\
&= -\boldsymbol{\epsilon}^{-1} \cdot \left[\underline{\nabla} \wedge \left(\vec{\vec{v}} \cdot \boldsymbol{\epsilon}\right)\right] && \text{(A5.5)}\\
&= -\boldsymbol{\epsilon}^{-1} \cdot \left\{\underline{\nabla} \wedge \left[\left(\frac{1}{2} v^{\beta\gamma} \vec{e}_\beta \wedge \vec{e}_\gamma\right) \cdot \left(\epsilon_{1\cdots N}\, \underline{e}^1 \wedge \ldots \wedge \underline{e}^N\right)\right]\right\} && \text{(A5.6)}\\
&= -\frac{1}{2} \boldsymbol{\epsilon}^{-1} \cdot \left\{\left[\frac{\partial}{\partial x^\alpha}\left(\epsilon_{1\cdots N} v^{\beta\gamma}\right)\right] \underline{e}^\alpha \wedge \left[\left(\vec{e}_\beta \wedge \vec{e}_\gamma\right) \cdot \left(\underline{e}^1 \wedge \ldots \wedge \underline{e}^N\right)\right]\right\} && \text{(A5.7)}\\
&= -\frac{1}{2}\left[\frac{1}{\epsilon_{1\cdots N}} \frac{\partial}{\partial x^\alpha}\left(\epsilon_{1\cdots N} v^{\beta\gamma}\right)\right] \boldsymbol{\epsilon}^{-1} \cdot \left\{\underline{e}^\alpha \wedge \left[\left(\vec{e}_\beta \wedge \vec{e}_\gamma\right) \cdot \boldsymbol{\epsilon}\right]\right\} && \text{(A5.8)}\\
&= -\frac{1}{2}\left[\frac{1}{\epsilon_{1\cdots N}} \frac{\partial}{\partial x^\alpha}\left(\epsilon_{1\cdots N} v^{\beta\gamma}\right)\right] \star^{-1} \left[\underline{e}^\alpha \wedge \star\left(\vec{e}_\beta \wedge \vec{e}_\gamma\right)\right] && \text{(A5.9)}\\
&= -\frac{1}{2}\left[\frac{1}{\epsilon_{1\cdots N}} \frac{\partial}{\partial x^\alpha}\left(\epsilon_{1\cdots N} v^{\beta\gamma}\right)\right] \left(\vec{e}_\beta \wedge \vec{e}_\gamma\right) \cdot \underline{e}^\alpha && \text{(A5.10)}\\
&= -\frac{1}{2}\left[\frac{1}{\epsilon_{1\cdots N}} \frac{\partial}{\partial x^\alpha}\left(\epsilon_{1\cdots N} v^{\beta\gamma}\right)\right] \left(\delta^\alpha_\gamma \vec{e}_\beta - \delta^\alpha_\beta \vec{e}_\gamma\right) && \text{(A5.11)}\\
&= \left[\frac{1}{\epsilon_{1\cdots N}} \frac{\partial}{\partial x^\alpha}\left(\epsilon_{1\cdots N} v^{\alpha\beta}\right)\right] \vec{e}_\beta && \text{(A5.12)}
\end{aligned}$$

therefore

$$\underline{\nabla} \cdot \left(\diamond\, \underline{\nabla} \wedge \diamond\vec{v}\right) = \frac{\vec{e}_\beta}{\epsilon_{1\cdots N}} \frac{\partial}{\partial x^\alpha}\left[\epsilon_{1\cdots N}\, g^{[\alpha\beta][\gamma\delta]} \frac{\partial}{\partial x^\gamma}\left(g_{\delta\epsilon} v^\epsilon\right)\right] \qquad \text{(A5.13)}$$

In polar coordinates

$$\begin{aligned}
g_{rr} &= 1 \quad , \quad g_{\theta\theta} = r^2\\
g^{rr} &= 1 \quad , \quad g^{\theta\theta} = \frac{1}{r^2}
\end{aligned} \qquad \text{(A5.14)}$$

$$\epsilon_{r\theta} = r \quad , \quad g^{[r\theta][r\theta]} = \frac{1}{r^2} \qquad \text{(A5.15)}$$

and the curvature is zero. Therefore, raising the index in Eq. (A10.3.13),

$$\nabla^2 \vec{v} = -\triangle \vec{v} \qquad \text{(A5.16)}$$

and

$$\begin{aligned}
\diamond\underline{\nabla} \wedge \diamond\left(\underline{\nabla} \cdot \vec{v}\right) &= \vec{e}_\alpha g^{\alpha\beta} \frac{\partial}{\partial x^\beta}\left[\frac{1}{\epsilon_{1\cdots N}} \frac{\partial}{\partial x^\gamma}\left(\epsilon_{1\cdots N} v^\gamma\right)\right] && \text{(A5.17)}\\
&= \left(\vec{e}_r g^{rr} \frac{\partial}{\partial r} + \vec{e}_\theta g^{\theta\theta} \frac{\partial}{\partial \theta}\right)\left[\frac{1}{\epsilon_{r\theta}} \frac{\partial}{\partial r}\left(\epsilon_{r\theta} v^r\right) + \frac{1}{\epsilon_{r\theta}} \frac{\partial}{\partial \theta}\left(\epsilon_{r\theta} v^\theta\right)\right] && \text{(A5.18)}\\
&= \left(\vec{e}_r \frac{\partial}{\partial r} + \frac{\vec{e}_\theta}{r^2} \frac{\partial}{\partial \theta}\right)\left[\frac{1}{r} \frac{\partial}{\partial r}\left(r v^r\right) + \frac{\partial v^\theta}{\partial \theta}\right] && \text{(A5.19)}\\
&= \vec{e}_r \frac{\partial}{\partial r}\left[\frac{1}{r} \frac{\partial}{\partial r}\left(r v^r\right)\right] + \vec{e}_r \frac{\partial^2 v^\theta}{\partial r \partial \theta} + \frac{\vec{e}_\theta}{r^3} \frac{\partial}{\partial r}\left(r \frac{\partial v^r}{\partial \theta}\right) + \frac{\vec{e}_\theta}{r^2} \frac{\partial^2 v^\theta}{\partial \theta^2} && \text{(A5.20)}
\end{aligned}$$

and

$$\underline{\nabla} \cdot (\diamond \underline{\nabla} \wedge \diamond \vec{v}) = \frac{\vec{e}_\beta}{\epsilon_{1\cdots N}} \frac{\partial}{\partial x^\alpha} \left[\epsilon_{1\cdots N} g^{[\alpha\beta][\gamma\delta]} \frac{\partial}{\partial x^\gamma} \left(g_{\delta\epsilon} v^\epsilon \right) \right] \tag{A5.21}$$

$$= \left(\frac{\vec{e}_\theta}{\epsilon_{r\theta}} \frac{\partial}{\partial r} - \frac{\vec{e}_r}{\epsilon_{r\theta}} \frac{\partial}{\partial \theta} \right) \left\{ \epsilon_{r\theta} g^{[r\theta][r\theta]} \left[\frac{\partial}{\partial r} \left(g_{\theta\theta} v^\theta \right) - \frac{\partial}{\partial \theta} \left(g_{rr} v^r \right) \right] \right\} \tag{A5.22}$$

$$= \left(\frac{\vec{e}_\theta}{r} \frac{\partial}{\partial r} - \frac{\vec{e}_r}{r} \frac{\partial}{\partial \theta} \right) \left\{ \frac{1}{r} \left[\frac{\partial}{\partial r} \left(r^2 v^\theta \right) - \frac{\partial v^r}{\partial \theta} \right] \right\} \tag{A5.23}$$

$$= \frac{\vec{e}_\theta}{r} \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r^2 v^\theta \right) \right] - \frac{\vec{e}_\theta}{r} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial v^r}{\partial \theta} \right) - \frac{\vec{e}_r}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v^\theta}{\partial \theta} \right) + \frac{\vec{e}_r}{r^2} \frac{\partial^2 v^r}{\partial \theta^2} \tag{A5.24}$$

The unit basis vectors $\vec{e}_{\hat{\alpha}}$ and components $v^{\hat{\alpha}}$ are related to the coordinate basis vectors $\vec{e}_\alpha$ and components v^α by

$$\begin{aligned} \vec{e}_r &= \vec{e}_{\hat{r}} \quad , \quad \vec{e}_\theta = r \vec{e}_{\hat{\theta}} \\ v^r &= v^{\hat{r}} \quad , \quad v^\theta = \frac{v^{\hat{\theta}}}{r} \end{aligned} \tag{A5.25}$$

therefore

$$\nabla^2 \vec{v} = \diamond \underline{\nabla} \wedge \diamond (\underline{\nabla} \cdot \vec{v}) + \underline{\nabla} \cdot (\diamond \underline{\nabla} \wedge \diamond \vec{v}) \tag{A5.26}$$

$$\begin{aligned} &= \vec{e}_r \left\{ \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r v^r \right) \right] + \frac{1}{r^2} \frac{\partial^2 v^r}{\partial \theta^2} - \frac{2}{r} \frac{\partial v^\theta}{\partial \theta} \right\} \\ &\quad + \frac{\vec{e}_\theta}{r} \left\{ \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r^2 v^\theta \right) \right] + \frac{1}{r} \frac{\partial^2 v^\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v^r}{\partial \theta} \right\} \end{aligned} \tag{A5.27}$$

$$\begin{aligned} &= \vec{e}_{\hat{r}} \left\{ \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r v^{\hat{r}} \right) \right] + \frac{1}{r^2} \frac{\partial^2 v^{\hat{r}}}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v^{\hat{\theta}}}{\partial \theta} \right\} \\ &\quad + \vec{e}_{\hat{\theta}} \left\{ \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r v^{\hat{\theta}} \right) \right] + \frac{1}{r^2} \frac{\partial^2 v^{\hat{\theta}}}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v^{\hat{r}}}{\partial \theta} \right\} \end{aligned} \tag{A5.28}$$