Homework 3 - Topological tensor calculus

Q3.1. Show that the Lie derivative

$$\mathcal{L}_v \boldsymbol{\omega} = \vec{v} \cdot (\underline{\nabla} \wedge \boldsymbol{\omega}) + \underline{\nabla} \wedge (\vec{v} \cdot \boldsymbol{\omega}) \tag{Q3.1.1}$$

commutes with the exterior derivative and obeys the Leibniz rule

$$\mathcal{L}_v (\boldsymbol{\omega} \wedge \boldsymbol{\sigma}) = (\mathcal{L}_v \boldsymbol{\omega}) \wedge \boldsymbol{\sigma} + \boldsymbol{\omega} \wedge \mathcal{L}_v \boldsymbol{\sigma} \tag{Q3.1.2}$$

Q3.2. Consider a spacetime with time coordinate t, time covector

$$\underline{e}^t = \underline{\nabla} \wedge t \tag{Q3.2.1}$$

time vector $\vec{e}_t$ satisfying

$$\vec{e}_t \cdot \underline{e}^t = 1 \tag{Q3.2.2}$$

and time and spatial exterior derivatives

$$\dot{\boldsymbol{\omega}} \equiv \mathcal{L}_{e_t} \boldsymbol{\omega} = \vec{e}_t \cdot (\underline{\nabla} \wedge \boldsymbol{\omega}) + \underline{\nabla} \wedge (\vec{e}_t \cdot \boldsymbol{\omega}) \tag{Q3.2.3}$$

$$\underline{\nabla}^{(3)} \wedge \boldsymbol{\omega} \equiv \vec{e}_t \cdot \left(\underline{e}^t \wedge \underline{\nabla} \wedge \boldsymbol{\omega} \right) - \underline{e}^t \wedge \underline{\nabla} \wedge (\vec{e}_t \cdot \boldsymbol{\omega}) \tag{Q3.2.4}$$

Show that

(a) the spacetime exterior derivative decomposes as

$$\underline{\nabla} \wedge \boldsymbol{\omega} = \underline{e}^t \wedge \dot{\boldsymbol{\omega}} + \underline{\nabla}^{(3)} \wedge \boldsymbol{\omega} \tag{Q3.2.5}$$

(b) the time and spatial exterior derivatives commute,

(c) the spatial exterior derivative obeys

$$\underline{\nabla}^{(3)} \wedge \underline{\nabla}^{(3)} \wedge \boldsymbol{\omega} = 0 \tag{Q3.2.6}$$

and

$$\underline{\nabla}^{(3)} \wedge (\boldsymbol{\omega} \wedge \boldsymbol{\sigma}) = \left(\underline{\nabla}^{(3)} \wedge \boldsymbol{\omega} \right) \wedge \boldsymbol{\sigma} + (-1)^{\deg \boldsymbol{\omega}} \boldsymbol{\omega} \wedge \left(\underline{\nabla}^{(3)} \wedge \boldsymbol{\sigma} \right) \tag{Q3.2.7}$$

as required for an exterior derivative.