Homework 3 - Topological tensor calculus

Q3.1. Show that the Lie derivative

$$\mathcal{L}_v\boldsymbol{\omega} = \vec{v}\cdot(\underline{\nabla}\wedge\boldsymbol{\omega}) + \underline{\nabla}\wedge(\vec{v}\cdot\boldsymbol{\omega}) \tag{Q3.1.1}$$

commutes with the exterior derivative and obeys the Leibnitz rule

$$\mathcal{L}_v(\boldsymbol{\omega}\wedge\boldsymbol{\sigma}) = (\mathcal{L}_v\boldsymbol{\omega})\wedge\boldsymbol{\sigma} + \boldsymbol{\omega}\wedge\mathcal{L}_v\boldsymbol{\sigma} \tag{Q3.1.2}$$

A3.1. Using Eq. (1.2.4),

$$\begin{aligned}
\underline{\nabla}\wedge\mathcal{L}_v\boldsymbol{\omega} &= \underline{\nabla}\wedge[\vec{v}\cdot(\underline{\nabla}\wedge\boldsymbol{\omega})] + \underline{\nabla}\wedge[\underline{\nabla}\wedge(\vec{v}\cdot\boldsymbol{\omega})] && \text{(A3.1.1)}\\
&= \vec{v}\cdot[\underline{\nabla}\wedge(\underline{\nabla}\wedge\boldsymbol{\omega})] + \underline{\nabla}\wedge[\vec{v}\cdot(\underline{\nabla}\wedge\boldsymbol{\omega})] && \text{(A3.1.2)}\\
&= \mathcal{L}_v(\underline{\nabla}\wedge\boldsymbol{\omega}) && \text{(A3.1.3)}
\end{aligned}$$

and using Eqs. (1.2.5) and (1.1.22),

$$\begin{aligned}
\mathcal{L}_v(\boldsymbol{\omega}\wedge\boldsymbol{\sigma}) &= \vec{v}\cdot[\underline{\nabla}\wedge(\boldsymbol{\omega}\wedge\boldsymbol{\sigma})] + \underline{\nabla}\wedge[\vec{v}\cdot(\boldsymbol{\omega}\wedge\boldsymbol{\sigma})] && \text{(A3.1.4)}\\
&= \vec{v}\cdot[(\underline{\nabla}\wedge\boldsymbol{\omega})\wedge\boldsymbol{\sigma}] + (-1)^{\deg\boldsymbol{\omega}}\vec{v}\cdot[\boldsymbol{\omega}\wedge(\underline{\nabla}\wedge\boldsymbol{\sigma})]\\
&\quad + \underline{\nabla}\wedge[(\vec{v}\cdot\boldsymbol{\omega})\wedge\boldsymbol{\sigma}] + (-1)^{\deg\boldsymbol{\omega}}\underline{\nabla}\wedge[\boldsymbol{\omega}\wedge(\vec{v}\cdot\boldsymbol{\sigma})] && \text{(A3.1.5)}\\
&= [\vec{v}\cdot(\underline{\nabla}\wedge\boldsymbol{\omega})]\wedge\boldsymbol{\sigma} + (-1)^{\deg\boldsymbol{\omega}+1}(\underline{\nabla}\wedge\boldsymbol{\omega})\wedge(\vec{v}\cdot\boldsymbol{\sigma})\\
&\quad + (-1)^{\deg\boldsymbol{\omega}}(\vec{v}\cdot\boldsymbol{\omega})\wedge(\underline{\nabla}\wedge\boldsymbol{\sigma}) + \boldsymbol{\omega}\wedge[\vec{v}\cdot(\underline{\nabla}\wedge\boldsymbol{\sigma})]\\
&\quad + [\underline{\nabla}\wedge(\vec{v}\cdot\boldsymbol{\omega})]\wedge\boldsymbol{\sigma} + (-1)^{\deg\boldsymbol{\omega}-1}(\vec{v}\cdot\boldsymbol{\omega})\wedge(\underline{\nabla}\wedge\boldsymbol{\sigma})\\
&\quad + (-1)^{\deg\boldsymbol{\omega}}(\underline{\nabla}\wedge\boldsymbol{\omega})\wedge(\vec{v}\cdot\boldsymbol{\sigma}) + \boldsymbol{\omega}\wedge[\underline{\nabla}\wedge(\vec{v}\cdot\boldsymbol{\sigma})] && \text{(A3.1.6)}\\
&= (\mathcal{L}_v\boldsymbol{\omega})\wedge\boldsymbol{\sigma} + \boldsymbol{\omega}\wedge\mathcal{L}_v\boldsymbol{\sigma} && \text{(A3.1.7)}
\end{aligned}$$

Q3.2. Consider a spacetime with time coordinate t, time covector

$$\underline{e}^t = \underline{\nabla}\wedge t \tag{Q3.2.1}$$

time vector $\vec{e}_t$ satisfying

$$\vec{e}_t\cdot\underline{e}^t = 1 \tag{Q3.2.2}$$

and time and spatial exterior derivatives

$$\begin{aligned}
\dot{\boldsymbol{\omega}} &\equiv \mathcal{L}_{e_t}\boldsymbol{\omega} = \vec{e}_t\cdot(\underline{\nabla}\wedge\boldsymbol{\omega}) + \underline{\nabla}\wedge(\vec{e}_t\cdot\boldsymbol{\omega}) && \text{(Q3.2.3)}\\
\underline{\nabla}^{(3)}\wedge\boldsymbol{\omega} &\equiv \vec{e}_t\cdot\left(\underline{e}^t\wedge\underline{\nabla}\wedge\boldsymbol{\omega}\right) - \underline{e}^t\wedge\underline{\nabla}\wedge(\vec{e}_t\cdot\boldsymbol{\omega}) && \text{(Q3.2.4)}
\end{aligned}$$

Show that

(a) the spacetime exterior derivative decomposes as

$$\underline{\nabla}\wedge\boldsymbol{\omega} = \underline{e}^t\wedge\dot{\boldsymbol{\omega}} + \underline{\nabla}^{(3)}\wedge\boldsymbol{\omega} \tag{Q3.2.5}$$

(b) the time and spatial exterior derivatives commute,

(c) the spatial exterior derivative obeys

$$\underline{\nabla}^{(3)} \wedge \underline{\nabla}^{(3)} \wedge \boldsymbol{\omega} = 0 \tag{Q3.2.6}$$

and

$$\underline{\nabla}^{(3)} \wedge (\boldsymbol{\omega} \wedge \boldsymbol{\sigma}) = \left(\underline{\nabla}^{(3)} \wedge \boldsymbol{\omega}\right) \wedge \boldsymbol{\sigma} + (-1)^{\deg \boldsymbol{\omega}} \boldsymbol{\omega} \wedge \left(\underline{\nabla}^{(3)} \wedge \boldsymbol{\sigma}\right) \tag{Q3.2.7}$$

as required for an exterior derivative.

A3.2. Eqs. (Q3.2.1) and (1.2.4) give

$$\underline{\nabla} \wedge \underline{e}^t = \underline{\nabla} \wedge \underline{\nabla} \wedge t = 0 \tag{A3.2.1}$$

Eqs. (1.1.22) and (Q3.2.2) give

$$\vec{e}_t \cdot \left(\underline{e}^t \wedge \underline{\nabla} \wedge \boldsymbol{\omega}\right) = \underline{\nabla} \wedge \boldsymbol{\omega} - \underline{e}^t \wedge [\vec{e}_t \cdot (\underline{\nabla} \wedge \boldsymbol{\omega})] \tag{A3.2.2}$$

and Eqs. (A3.2.2), (1.2.4), (1.2.5) and (A3.2.1) give

$$\begin{aligned} \underline{\nabla} \wedge \left[\vec{e}_t \cdot \left(\underline{e}^t \wedge \underline{\nabla} \wedge \boldsymbol{\omega}\right)\right] &= -\underline{\nabla} \wedge \left\{\underline{e}^t \wedge [\vec{e}_t \cdot (\underline{\nabla} \wedge \boldsymbol{\omega})]\right\} && \text{(A3.2.3)} \\ &= \underline{e}^t \wedge \underline{\nabla} \wedge [\vec{e}_t \cdot (\underline{\nabla} \wedge \boldsymbol{\omega})] && \text{(A3.2.4)} \end{aligned}$$

(a) Using Eqs. (A3.2.2), (Q3.2.3) and (Q3.2.4),

$$\begin{aligned} \underline{\nabla} \wedge \boldsymbol{\omega} &= \underline{e}^t \wedge [\vec{e}_t \cdot (\underline{\nabla} \wedge \boldsymbol{\omega})] + \vec{e}_t \cdot \left(\underline{e}^t \wedge \underline{\nabla} \wedge \boldsymbol{\omega}\right) && \text{(A3.2.5)} \\ &= \underline{e}^t \wedge [\vec{e}_t \cdot (\underline{\nabla} \wedge \boldsymbol{\omega}) + \underline{\nabla} \wedge (\vec{e}_t \cdot \boldsymbol{\omega})] \\ &\quad + \vec{e}_t \cdot \left(\underline{e}^t \wedge \underline{\nabla} \wedge \boldsymbol{\omega}\right) - \underline{e}^t \wedge \underline{\nabla} \wedge (\vec{e}_t \cdot \boldsymbol{\omega}) && \text{(A3.2.6)} \\ &= \underline{e}^t \wedge \dot{\boldsymbol{\omega}} + \underline{\nabla}^{(3)} \wedge \boldsymbol{\omega} && \text{(A3.2.7)} \end{aligned}$$

(b) Using Eqs. (Q3.2.3), (Q3.2.4), (A3.2.4), (1.2.5) and (A3.2.1),

$$\begin{aligned} &\mathcal{L}_{e_t} \left(\underline{\nabla}^{(3)} \wedge \boldsymbol{\omega}\right) \\ &= \vec{e}_t \cdot \left\{\underline{\nabla} \wedge \left[\vec{e}_t \cdot \left(\underline{e}^t \wedge \underline{\nabla} \wedge \boldsymbol{\omega}\right)\right]\right\} + \underline{\nabla} \wedge \left\{\vec{e}_t \cdot \left[\vec{e}_t \cdot \left(\underline{e}^t \wedge \underline{\nabla} \wedge \boldsymbol{\omega}\right)\right]\right\} \\ &\quad - \vec{e}_t \cdot \left\{\underline{\nabla} \wedge \left[\underline{e}^t \wedge \underline{\nabla} \wedge (\vec{e}_t \cdot \boldsymbol{\omega})\right]\right\} - \underline{\nabla} \wedge \left\{\vec{e}_t \cdot \left[\underline{e}^t \wedge \underline{\nabla} \wedge (\vec{e}_t \cdot \boldsymbol{\omega})\right]\right\} && \text{(A3.2.8)} \\ &= \vec{e}_t \cdot \left\{\underline{e}^t \wedge \underline{\nabla} \wedge [\vec{e}_t \cdot (\underline{\nabla} \wedge \boldsymbol{\omega})]\right\} - \underline{e}^t \wedge \underline{\nabla} \wedge \{\vec{e}_t \cdot [\vec{e}_t \cdot (\underline{\nabla} \wedge \boldsymbol{\omega})]\} \\ &\quad + \vec{e}_t \cdot \left\{\underline{e}^t \wedge \underline{\nabla} \wedge [\underline{\nabla} \wedge (\vec{e}_t \cdot \boldsymbol{\omega})]\right\} - \underline{e}^t \wedge \underline{\nabla} \wedge \{\vec{e}_t \cdot [\underline{\nabla} \wedge (\vec{e}_t \cdot \boldsymbol{\omega})]\} && \text{(A3.2.9)} \\ &= \underline{\nabla}^{(3)} \wedge \mathcal{L}_{e_t} \boldsymbol{\omega} && \text{(A3.2.10)} \end{aligned}$$

(c) Using Eqs. (Q3.2.4), (A3.2.4), (1.2.5) and (A3.2.1),

$$\begin{aligned} &\underline{\nabla}^{(3)} \wedge \underline{\nabla}^{(3)} \wedge \boldsymbol{\omega} \\ &= \vec{e}_t \cdot \left\{\underline{e}^t \wedge \underline{\nabla} \wedge \left[\vec{e}_t \cdot \left(\underline{e}^t \wedge \underline{\nabla} \wedge \boldsymbol{\omega}\right)\right]\right\} - \underline{e}^t \wedge \underline{\nabla} \wedge \left\{\vec{e}_t \cdot \left[\vec{e}_t \cdot \left(\underline{e}^t \wedge \underline{\nabla} \wedge \boldsymbol{\omega}\right)\right]\right\} \\ &\quad - \vec{e}_t \cdot \left\{\underline{e}^t \wedge \underline{\nabla} \wedge \left[\underline{e}^t \wedge \underline{\nabla} \wedge (\vec{e}_t \cdot \boldsymbol{\omega})\right]\right\} + \underline{e}^t \wedge \underline{\nabla} \wedge \left\{\vec{e}_t \cdot \left[\underline{e}^t \wedge \underline{\nabla} \wedge (\vec{e}_t \cdot \boldsymbol{\omega})\right]\right\} && \text{(A3.2.11)} \\ &= \vec{e}_t \cdot \left\{\underline{e}^t \wedge \underline{e}^t \wedge \underline{\nabla} \wedge [\vec{e}_t \cdot (\underline{\nabla} \wedge \boldsymbol{\omega})]\right\} - \underline{e}^t \wedge \underline{\nabla} \wedge \left\{\vec{e}_t \cdot \left[\vec{e}_t \cdot \left(\underline{e}^t \wedge \underline{\nabla} \wedge \boldsymbol{\omega}\right)\right]\right\} \\ &\quad + \vec{e}_t \cdot \left[\underline{e}^t \wedge \underline{e}^t \wedge \underline{\nabla} \wedge \underline{\nabla} \wedge (\vec{e}_t \cdot \boldsymbol{\omega})\right] + \underline{e}^t \wedge \underline{e}^t \wedge \underline{\nabla} \wedge \{\vec{e}_t \cdot [\underline{\nabla} \wedge (\vec{e}_t \cdot \boldsymbol{\omega})]\} && \text{(A3.2.12)} \\ &= 0 && \text{(A3.2.13)} \end{aligned}$$

and, using Eqs. (Q3.2.5), (1.2.5), (Q3.1.2) and (1.1.20),

$$\begin{aligned}\underline{\nabla}^{(3)} \wedge (\boldsymbol{\omega} \wedge \boldsymbol{\sigma}) &= \underline{\nabla} \wedge (\boldsymbol{\omega} \wedge \boldsymbol{\sigma}) - \underline{e}^t \wedge \mathcal{L}_{e_t} (\boldsymbol{\omega} \wedge \boldsymbol{\sigma}) \qquad (A3.2.14)\\ &= (\underline{\nabla} \wedge \boldsymbol{\omega}) \wedge \boldsymbol{\sigma} + (-1)^{\deg \boldsymbol{\omega}} \boldsymbol{\omega} \wedge (\underline{\nabla} \wedge \boldsymbol{\sigma}) \\ &\quad - \left(\underline{e}^t \wedge \mathcal{L}_{e_t} \boldsymbol{\omega}\right) \wedge \boldsymbol{\sigma} - (-1)^{\deg \boldsymbol{\omega}} \boldsymbol{\omega} \wedge \left(\underline{e}^t \wedge \mathcal{L}_{e_t} \boldsymbol{\sigma}\right) \qquad (A3.2.15)\\ &= \left(\underline{\nabla}^{(3)} \wedge \boldsymbol{\omega}\right) \wedge \boldsymbol{\sigma} + (-1)^{\deg \boldsymbol{\omega}} \boldsymbol{\omega} \wedge \left(\underline{\nabla}^{(3)} \wedge \boldsymbol{\sigma}\right) \qquad (A3.2.16)\end{aligned}$$

or alternatively, using Eqs. (Q3.2.4), (1.2.5), (1.1.20) and (1.1.22),

$$\begin{aligned}&\underline{\nabla}^{(3)} \wedge (\boldsymbol{\omega} \wedge \boldsymbol{\sigma}) \\ &= \vec{e}_t \cdot \left[\underline{e}^t \wedge \underline{\nabla} \wedge (\boldsymbol{\omega} \wedge \boldsymbol{\sigma})\right] - \underline{e}^t \wedge \underline{\nabla} \wedge \left[\vec{e}_t \cdot (\boldsymbol{\omega} \wedge \boldsymbol{\sigma})\right] \qquad (A3.2.17)\\ &= \vec{e}_t \cdot \left[\left(\underline{e}^t \wedge \underline{\nabla} \wedge \boldsymbol{\omega}\right) \wedge \boldsymbol{\sigma}\right] + \vec{e}_t \cdot \left[\boldsymbol{\omega} \wedge \left(\underline{e}^t \wedge \underline{\nabla} \wedge \boldsymbol{\sigma}\right)\right] \\ &\quad - \underline{e}^t \wedge \underline{\nabla} \wedge \left[(\vec{e}_t \cdot \boldsymbol{\omega}) \wedge \boldsymbol{\sigma}\right] - (-1)^{\deg \boldsymbol{\omega}} \underline{e}^t \wedge \underline{\nabla} \wedge \left[\boldsymbol{\omega} \wedge (\vec{e}_t \cdot \boldsymbol{\sigma})\right] \qquad (A3.2.18)\\ &= \left[\vec{e}_t \cdot \left(\underline{e}^t \wedge \underline{\nabla} \wedge \boldsymbol{\omega}\right)\right] \wedge \boldsymbol{\sigma} + (-1)^{\deg \boldsymbol{\omega}+2} \left(\underline{e}^t \wedge \underline{\nabla} \wedge \boldsymbol{\omega}\right) \wedge (\vec{e}_t \cdot \boldsymbol{\sigma}) \\ &\quad + (\vec{e}_t \cdot \boldsymbol{\omega}) \wedge \left(\underline{e}^t \wedge \underline{\nabla} \wedge \boldsymbol{\sigma}\right) + (-1)^{\deg \boldsymbol{\omega}} \boldsymbol{\omega} \wedge \left[\vec{e}_t \cdot \left(\underline{e}^t \wedge \underline{\nabla} \wedge \boldsymbol{\sigma}\right)\right] \\ &\quad - \left[\underline{e}^t \wedge \underline{\nabla} \wedge (\vec{e}_t \cdot \boldsymbol{\omega})\right] \wedge \boldsymbol{\sigma} - (\vec{e}_t \cdot \boldsymbol{\omega}) \wedge \left(\underline{e}^t \wedge \underline{\nabla} \wedge \boldsymbol{\sigma}\right) \\ &\quad - (-1)^{\deg \boldsymbol{\omega}} \left(\underline{e}^t \wedge \underline{\nabla} \wedge \boldsymbol{\omega}\right) \wedge (\vec{e}_t \cdot \boldsymbol{\sigma}) - (-1)^{\deg \boldsymbol{\omega}} \boldsymbol{\omega} \wedge \left[\underline{e}^t \wedge \underline{\nabla} \wedge (\vec{e}_t \cdot \boldsymbol{\sigma})\right] \qquad (A3.2.19)\\ &= \left(\underline{\nabla}^{(3)} \wedge \boldsymbol{\omega}\right) \wedge \boldsymbol{\sigma} + (-1)^{\deg \boldsymbol{\omega}} \boldsymbol{\omega} \wedge \left(\underline{\nabla}^{(3)} \wedge \boldsymbol{\sigma}\right) \qquad (A3.2.20)\end{aligned}$$