Homework 5 - Densities and volumes

Q5.1. Explain the conceptual difference between $\underset{\equiv}{\rho}$ and $\rho = \star\underset{\equiv}{\rho}$ in three dimensions.

A5.1. $\underset{\equiv}{\rho}$ is a density and $\rho = \star\underset{\equiv}{\rho} = \vec{\vec{\vec{\epsilon}}} \cdot \underset{\equiv}{\rho}$ combines the unit volume $\vec{\vec{\vec{\epsilon}}}$ with the density $\underset{\equiv}{\rho}$ to give the traditional scalar representation of a density: amount per unit volume.

Q5.2. Show that

(a)

$$\boldsymbol{\epsilon}^{-1} \cdot (\vec{v} \cdot \boldsymbol{\epsilon}) = \vec{v} \tag{Q5.2.1}$$

(b)

$$\boldsymbol{\epsilon}^{-1} \cdot [(\vec{u} \wedge \vec{v}) \cdot \boldsymbol{\epsilon}] = \vec{u} \wedge \vec{v} \tag{Q5.2.2}$$

A5.2. (a) Using Eqs. (1.1.21), (1.1.22), (1.4.1) and

$$\vec{v} \wedge \boldsymbol{\epsilon}^{-1} = 0 \tag{A5.2.1}$$

we have

$$\begin{aligned}
\boldsymbol{\epsilon}^{-1} \cdot (\vec{v} \cdot \boldsymbol{\epsilon}) &= -(-1)^N \boldsymbol{\epsilon}^{-1} \cdot (\boldsymbol{\epsilon} \cdot \vec{v}) && \text{(A5.2.2)} \\
&= -\left(\vec{v} \wedge \boldsymbol{\epsilon}^{-1}\right) \cdot \boldsymbol{\epsilon} + \vec{v}\left(\boldsymbol{\epsilon}^{-1} \cdot \boldsymbol{\epsilon}\right) && \text{(A5.2.3)} \\
&= \vec{v} && \text{(A5.2.4)}
\end{aligned}$$

(b) Similarly

$$\begin{aligned}
\boldsymbol{\epsilon}^{-1} \cdot [(\vec{u} \wedge \vec{v}) \cdot \boldsymbol{\epsilon}] &= \boldsymbol{\epsilon}^{-1} \cdot [\vec{v} \cdot (\vec{u} \cdot \boldsymbol{\epsilon})] && \text{(A5.2.5)} \\
&= (-1)^N \boldsymbol{\epsilon}^{-1} \cdot [(\vec{u} \cdot \boldsymbol{\epsilon}) \cdot \vec{v}] && \text{(A5.2.6)} \\
&= \left(\vec{v} \wedge \boldsymbol{\epsilon}^{-1}\right) \cdot (\vec{u} \cdot \boldsymbol{\epsilon}) - \vec{v} \wedge \left[\boldsymbol{\epsilon}^{-1} \cdot (\vec{u} \cdot \boldsymbol{\epsilon})\right] && \text{(A5.2.7)} \\
&= -\vec{v} \wedge \vec{u} && \text{(A5.2.8)} \\
&= \vec{u} \wedge \vec{v} && \text{(A5.2.9)}
\end{aligned}$$

Q5.3. Show that for an m-form $\boldsymbol{\omega}$ and an n-vector $\boldsymbol{v}$ with $m \leq n$

(a)

$$(\star\boldsymbol{\omega}) \cdot (\star\boldsymbol{v}) = \boldsymbol{\omega} \cdot \boldsymbol{v} \tag{Q5.3.1}$$

(b)

$$\star^{-1}(\boldsymbol{\omega} \wedge \star\boldsymbol{v}) = \boldsymbol{v} \cdot \boldsymbol{\omega} \tag{Q5.3.2}$$

A5.3. (a) Using Eqs. (1.4.4) and (1.1.19), for $m + (N - n) \leq N$,

$$(\star\boldsymbol{\omega}) \cdot (\star\boldsymbol{v}) = \left(\boldsymbol{\omega} \cdot \boldsymbol{\epsilon}^{-1}\right) \cdot \star\boldsymbol{v} = \boldsymbol{\omega} \cdot \left(\boldsymbol{\epsilon}^{-1} \cdot \star\boldsymbol{v}\right) = \boldsymbol{\omega} \cdot \boldsymbol{v} \tag{A5.3.1}$$

(b) Using Eqs. (1.4.5), (1.1.22) and (1.4.4), for $m + (N - n) \leq N$,

$$\star^{-1}(\boldsymbol{\omega} \wedge \star \boldsymbol{v}) = \boldsymbol{\epsilon}^{-1} \cdot (\boldsymbol{\omega} \wedge \star \boldsymbol{v}) = \left(\boldsymbol{\epsilon}^{-1} \cdot \star \boldsymbol{v}\right) \cdot \boldsymbol{\omega} = \boldsymbol{v} \cdot \boldsymbol{\omega} \tag{A5.3.2}$$

Q5.4. Show that

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 \tag{Q5.4.1}$$

for any multivector $\boldsymbol{v}$. What traditional vector calculus results does this correspond to?

A5.4. Eq. (1.4.7) for an n-vector $\boldsymbol{v}$ is

$$(-1)^{n-1} \boldsymbol{\nabla} \cdot \boldsymbol{v} = \star^{-1} \boldsymbol{\nabla} \wedge \star \boldsymbol{v} \tag{A5.4.1}$$

therefore Eq. (1.2.4) gives

$$-\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{v} = \star^{-1} \boldsymbol{\nabla} \wedge \boldsymbol{\nabla} \wedge \star \boldsymbol{v} = 0 \tag{A5.4.2}$$

For $\boldsymbol{v}$ a scalar or a vector, Eq. (Q5.4.1) is trivial. For $\boldsymbol{v}$ a two-vector, Eq. (Q5.4.1) can be written as

$$0 = \underline{\nabla} \cdot \underline{\nabla} \cdot \vec{\vec{v}} = -\underline{\nabla} \cdot \star^{-1} \underline{\nabla} \wedge \star \vec{\vec{v}} \tag{A5.4.3}$$

which corresponds to $\vec{\nabla} \cdot \vec{\nabla} \times \vec{v} = 0$ in three-dimensional vector calculus. For $\boldsymbol{v}$ a three-vector, Eq. (Q5.4.1) can be written as

$$0 = \underline{\nabla} \cdot \underline{\nabla} \cdot \vec{\vec{\vec{v}}} = - \star^{-1} \underline{\nabla} \wedge \underline{\nabla} \star \vec{\vec{\vec{v}}} \tag{A5.4.4}$$

which corresponds to $\vec{\nabla} \times \vec{\nabla} v = 0$ in three-dimensional vector calculus.