# Homework 6 - Bases and coordinates

Q6.1. Write down Maxwell's equations in spherical polar coordinates.

A6.1. Using Eqs. (1.5.8) and (1.5.22), we can expand Maxwell's equations, Eq. (1.3.1), in a coordinate basis as

$$\frac{1}{2!}\frac{\partial B_{\beta\gamma}}{\partial x^\alpha}\,\underline{e}^\alpha\wedge\underline{e}^\beta\wedge\underline{e}^\gamma = 0 \qquad \text{(A6.1.1)}$$

$$\frac{\partial E_\beta}{\partial x^\alpha}\,\underline{e}^\alpha\wedge\underline{e}^\beta + \frac{1}{2!}\frac{\partial B_{\alpha\beta}}{\partial t}\,\underline{e}^\alpha\wedge\underline{e}^\beta = 0 \qquad \text{(A6.1.2)}$$

$$\frac{1}{2!}\frac{\partial D_{\beta\gamma}}{\partial x^\alpha}\,\underline{e}^\alpha\wedge\underline{e}^\beta\wedge\underline{e}^\gamma = \frac{1}{3!}\rho_{\alpha\beta\gamma}\,\underline{e}^\alpha\wedge\underline{e}^\beta\wedge\underline{e}^\gamma \qquad \text{(A6.1.3)}$$

$$\frac{\partial H_\beta}{\partial x^\alpha}\,\underline{e}^\alpha\wedge\underline{e}^\beta - \frac{1}{2!}\frac{\partial D_{\alpha\beta}}{\partial t}\,\underline{e}^\alpha\wedge\underline{e}^\beta = \frac{1}{2!}j_{\alpha\beta}\,\underline{e}^\alpha\wedge\underline{e}^\beta \qquad \text{(A6.1.4)}$$

where we have assumed that $\dot{\underline{e}}^\alpha = 0$ as is appropriate for a spherical polar coordinate basis. The independent components are

$$\frac{1}{2!}B_{[\beta\gamma,\alpha]} = 0 \qquad \text{(A6.1.5)}$$

$$E_{[\beta,\alpha]} + \frac{1}{2!}\dot{B}_{[\alpha\beta]} = 0 \qquad \text{(A6.1.6)}$$

$$\frac{1}{2!}D_{[\beta\gamma,\alpha]} = \frac{1}{3!}\rho_{[\alpha\beta\gamma]} \qquad \text{(A6.1.7)}$$

$$H_{[\beta,\alpha]} - \frac{1}{2!}\dot{D}_{[\alpha\beta]} = \frac{1}{2!}j_{[\alpha\beta]} \qquad \text{(A6.1.8)}$$

where $B_{\beta\gamma,\alpha} \equiv \partial B_{\beta\gamma}/\partial x^\alpha$ and square brackets denote antisymmetrization. More explicitly, in spherical polar coordinates

$$B_{\theta\phi,r} + B_{\phi r,\theta} + B_{r\theta,\phi} = 0 \qquad \text{(A6.1.9)}$$

$$E_{\theta,r} - E_{r,\theta} + \dot{B}_{r\theta} = 0 \qquad \text{(A6.1.10)}$$

$$E_{\phi,\theta} - E_{\theta,\phi} + \dot{B}_{\theta\phi} = 0 \qquad \text{(A6.1.11)}$$

$$E_{r,\phi} - E_{\phi,r} + \dot{B}_{\phi r} = 0 \qquad \text{(A6.1.12)}$$

$$D_{\theta\phi,r} + D_{\phi r,\theta} + D_{r\theta,\phi} = \rho_{r\theta\phi} \qquad \text{(A6.1.13)}$$

$$H_{\theta,r} - H_{r,\theta} - \dot{D}_{r\theta} = j_{r\theta} \qquad \text{(A6.1.14)}$$

$$H_{\phi,\theta} - H_{\theta,\phi} - \dot{D}_{\theta\phi} = j_{\theta\phi} \qquad \text{(A6.1.15)}$$

$$H_{r,\phi} - H_{\phi,r} - \dot{D}_{\phi r} = j_{\phi r} \qquad \text{(A6.1.16)}$$

Q6.2. Show that

$$\underline{\nabla}\cdot\vec{v} = \frac{1}{\epsilon_{1\cdots N}}\frac{\partial}{\partial x^\alpha}\left(\epsilon_{1\cdots N}v^\alpha\right) \qquad \text{(Q6.2.1)}$$

and hence write down $\underline{\nabla}\cdot\vec{v}$ in three dimensional spherical polar coordinates. Compare with the traditional vector calculus formula and explain the difference.

A6.2. Using Eqs. (1.4.7), (1.4.4) and (1.4.5), (1.5.2) and (1.5.13), (1.5.22), and (Q5.3.2),

$$\begin{aligned}
\underline{\nabla} \cdot \vec{v} &= \star^{-1} \underline{\nabla} \wedge \star \vec{v} && \text{(A6.2.1)} \\
&= \boldsymbol{\epsilon}^{-1} \cdot \left[ \underline{\nabla} \wedge (\vec{v} \cdot \boldsymbol{\epsilon}) \right] && \text{(A6.2.2)} \\
&= \boldsymbol{\epsilon}^{-1} \cdot \left\{ \underline{\nabla} \wedge \left[ \left( v^\beta \vec{e}_\beta \right) \cdot \left( \epsilon_{1\cdots N}\, \underline{e}^1 \wedge \ldots \wedge \underline{e}^N \right) \right] \right\} && \text{(A6.2.3)} \\
&= \boldsymbol{\epsilon}^{-1} \cdot \left\{ \underline{\nabla} \wedge \left[ \epsilon_{1\cdots N} v^\beta \vec{e}_\beta \cdot \left( \underline{e}^1 \wedge \ldots \wedge \underline{e}^N \right) \right] \right\} && \text{(A6.2.4)} \\
&= \boldsymbol{\epsilon}^{-1} \cdot \left\{ \left[ \frac{\partial}{\partial x^\alpha} \left( \epsilon_{1\cdots N} v^\beta \right) \right] \underline{e}^\alpha \wedge \left[ \vec{e}_\beta \cdot \left( \underline{e}^1 \wedge \ldots \wedge \underline{e}^N \right) \right] \right\} && \text{(A6.2.5)} \\
&= \left[ \frac{1}{\epsilon_{1\cdots N}} \frac{\partial}{\partial x^\alpha} \left( \epsilon_{1\cdots N} v^\beta \right) \right] \boldsymbol{\epsilon}^{-1} \cdot \left[ \underline{e}^\alpha \wedge (\vec{e}_\beta \cdot \boldsymbol{\epsilon}) \right] && \text{(A6.2.6)} \\
&= \left[ \frac{1}{\epsilon_{1\cdots N}} \frac{\partial}{\partial x^\alpha} \left( \epsilon_{1\cdots N} v^\beta \right) \right] \star^{-1} \left( \underline{e}^\alpha \wedge \star \vec{e}_\beta \right) && \text{(A6.2.7)} \\
&= \left[ \frac{1}{\epsilon_{1\cdots N}} \frac{\partial}{\partial x^\alpha} \left( \epsilon_{1\cdots N} v^\beta \right) \right] \vec{e}_\beta \cdot \underline{e}^\alpha && \text{(A6.2.8)} \\
&= \frac{1}{\epsilon_{1\cdots N}} \frac{\partial}{\partial x^\alpha} \left( \epsilon_{1\cdots N} v^\alpha \right) && \text{(A6.2.9)}
\end{aligned}$$

In spherical polar coordinates

$$\epsilon_{1\cdots N} = r^2 \sin\theta \qquad \text{(A6.2.10)}$$

therefore

$$\underline{\nabla} \cdot \vec{v} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 v^r \right) + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left( \sin\theta\, v^\theta \right) + \frac{\partial}{\partial\phi} \left( v^\phi \right) \qquad \text{(A6.2.11)}$$

The traditional vector calculus formula is

$$\vec{\nabla} \cdot \vec{v} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 v^{\hat{r}} \right) + \frac{1}{r\sin\theta} \frac{\partial}{\partial\theta} \left( \sin\theta\, v^{\hat{\theta}} \right) + \frac{1}{r\sin\theta} \frac{\partial}{\partial\phi} \left( v^{\hat{\phi}} \right) \qquad \text{(A6.2.12)}$$

The difference arises because traditional vector calculus uses unit basis vectors $\vec{e}_{\hat{\alpha}}$ rather than coordinate basis vectors $\vec{e}_\alpha$

$$\begin{aligned}
\vec{e}_{\hat{r}} &= \vec{e}_r && \text{(A6.2.13)} \\
\vec{e}_{\hat{\theta}} &= \frac{1}{r} \vec{e}_\theta && \text{(A6.2.14)} \\
\vec{e}_{\hat{\phi}} &= \frac{1}{r\sin\theta} \vec{e}_\phi && \text{(A6.2.15)}
\end{aligned}$$

and hence components

$$\begin{aligned}
v^{\hat{r}} &= v^r && \text{(A6.2.16)} \\
v^{\hat{\theta}} &= r v^\theta && \text{(A6.2.17)} \\
v^{\hat{\phi}} &= r\sin\theta\, v^\phi && \text{(A6.2.18)}
\end{aligned}$$