Homework 7 - Abstract index notation

Q7.1. Express Eqs (1.4.3), (1.4.6) and (1.4.8) in abstract index notation.

A7.1.

$$\rho = \frac{1}{3!}\epsilon^{[\mathbf{abc}]}\rho_{[\mathbf{abc}]} \tag{A7.1.1}$$

$$B^{\mathbf{a}} = \frac{1}{2!}\epsilon^{[\mathbf{abc}]}B_{[\mathbf{bc}]} \tag{A7.1.2}$$

$$\nabla_{\mathbf{a}}B^{\mathbf{a}} = \frac{1}{3!}\epsilon^{[\mathbf{abc}]}\left(\nabla_{\mathbf{a}}B_{[\mathbf{bc}]} + \nabla_{\mathbf{b}}B_{[\mathbf{ca}]} + \nabla_{\mathbf{c}}B_{[\mathbf{ab}]}\right) \tag{A7.1.3}$$

$$= \frac{1}{2}\epsilon^{[\mathbf{abc}]}\nabla_{\mathbf{a}}B_{[\mathbf{bc}]} \tag{A7.1.4}$$

Q7.2. Express Eqs. (1.5.2) and (1.5.3) in abstract index notation and show that

(a)

$$v^\alpha = e^\alpha_{\mathbf{a}}v^{\mathbf{a}} \tag{Q7.2.1}$$

(b)

$$\omega_{\mathbf{a}}v^{\mathbf{a}} = \omega_\alpha v^\alpha \tag{Q7.2.2}$$

(c)

$$e^\alpha_{\mathbf{a}}e^{\mathbf{b}}_\alpha = \delta^{\mathbf{b}}_{\mathbf{a}} \tag{Q7.2.3}$$

explaining the meaning of all terms.

A7.2. In abstract index notation, Eq. (1.5.2) is

$$v^{\mathbf{a}} = v^\alpha e^{\mathbf{a}}_\alpha \tag{A7.2.1}$$

where v^α is the α component of the vector $v^{\mathbf{a}}$, $e^{\mathbf{a}}_\alpha$ is the α basis vector and the right hand side is summed over the components α. Eq. (1.5.3) is

$$e^\alpha_{\mathbf{a}}e^{\mathbf{a}}_\beta = \delta^\alpha_\beta \tag{A7.2.2}$$

where $e^\alpha_{\mathbf{a}}e^{\mathbf{a}}_\beta$ is the α basis covector contracted with the β basis vector and the δ^α_β are the components of the identity tensor $\delta^{\mathbf{a}}_{\mathbf{b}}$, which are equal to 1 if $\alpha = \beta$ and zero otherwise.

(a)

$$e^\alpha_{\mathbf{a}}v^{\mathbf{a}} = e^\alpha_{\mathbf{a}}v^\beta e^{\mathbf{a}}_\beta = v^\beta\delta^\alpha_\beta = v^\alpha \tag{A7.2.3}$$

v^α is the α component of the vector $v^{\mathbf{a}}$ and $e^\alpha_{\mathbf{a}}v^{\mathbf{a}}$ is the α basis covector contracted with the vector $v^{\mathbf{a}}$.

(b)

$$\omega_{\mathbf{a}} v^{\mathbf{a}} = \omega_\alpha e^\alpha_{\mathbf{a}} v^\beta e^{\mathbf{a}}_\beta = \omega_\alpha v^\beta \delta^\alpha_\beta = \omega_\alpha v^\alpha \tag{A7.2.4}$$

$\omega_{\mathbf{a}} v^{\mathbf{a}}$ is the covector $\omega_{\mathbf{a}}$ contracted with the vector $v^{\mathbf{a}}$ and $\omega_\alpha v^\alpha$ is the sum of the products of the components of $\omega_{\mathbf{a}}$ and $v^{\mathbf{a}}$.

(c)

$$e^\alpha_{\mathbf{a}} e^{\mathbf{b}}_\alpha v^{\mathbf{a}} = e^{\mathbf{b}}_\alpha v^\alpha = v^{\mathbf{b}} \tag{A7.2.5}$$

therefore

$$e^\alpha_{\mathbf{a}} e^{\mathbf{b}}_\alpha = \delta^{\mathbf{b}}_{\mathbf{a}} \tag{A7.2.6}$$

$e^\alpha_{\mathbf{a}} e^{\mathbf{b}}_\alpha$ is the sum of the products of the basis covectors and vectors and $\delta^{\mathbf{b}}_{\mathbf{a}}$ is the identity tensor.

Q7.3. Using abstract index notation, show that

(a)

$$\vec{v} \cdot \left(\underline{\omega} \wedge \underline{\underline{\sigma}}\right) = \left(\vec{v} \cdot \underline{\omega}\right) \underline{\underline{\sigma}} + \left(\vec{v} \cdot \underline{\underline{\sigma}}\right) \wedge \underline{\omega} \tag{Q7.3.1}$$

(b)

$$\vec{\vec{v}} \cdot \left(\underline{\omega} \wedge \underline{\underline{\sigma}}\right) = \left(\underline{\omega} \cdot \vec{\vec{v}}\right) \cdot \underline{\underline{\sigma}} + \left(\vec{\vec{v}} \cdot \underline{\underline{\sigma}}\right) \underline{\omega} \tag{Q7.3.2}$$

A7.3. (a) Using Eqs. (2.1.13) and (2.1.12),

$$\begin{aligned}
\vec{v} \cdot \left(\underline{\omega} \wedge \underline{\underline{\sigma}}\right) &\leftrightarrow v^{\mathbf{a}} \left(\omega_{\mathbf{a}} \sigma_{[\mathbf{bc}]} + \omega_{\mathbf{b}} \sigma_{[\mathbf{ca}]} + \omega_{\mathbf{c}} \sigma_{[\mathbf{ab}]}\right) && \text{(A7.3.1)}\\
&= v^{\mathbf{a}} \omega_{\mathbf{a}} \sigma_{[\mathbf{bc}]} + v^{\mathbf{a}} \sigma_{[\mathbf{ab}]} \omega_{\mathbf{c}} + v^{\mathbf{a}} \sigma_{[\mathbf{ca}]} \omega_{\mathbf{b}} && \text{(A7.3.2)}\\
&= v^{\mathbf{a}} \omega_{\mathbf{a}} \sigma_{[\mathbf{bc}]} + v^{\mathbf{a}} \sigma_{[\mathbf{ab}]} \omega_{\mathbf{c}} - v^{\mathbf{a}} \sigma_{[\mathbf{ac}]} \omega_{\mathbf{b}} && \text{(A7.3.3)}\\
&\leftrightarrow \left(\vec{v} \cdot \underline{\omega}\right) \underline{\underline{\sigma}} + \left(\vec{v} \cdot \underline{\underline{\sigma}}\right) \wedge \underline{\omega} && \text{(A7.3.4)}
\end{aligned}$$

(b) Using Eqs. (2.1.11) and (2.1.13),

$$\begin{aligned}
\vec{\vec{v}} \cdot \left(\underline{\omega} \wedge \underline{\underline{\sigma}}\right) &\leftrightarrow \frac{1}{2!} v^{[\mathbf{ab}]} \left(\omega_{\mathbf{a}} \sigma_{[\mathbf{bc}]} + \omega_{\mathbf{b}} \sigma_{[\mathbf{ca}]} + \omega_{\mathbf{c}} \sigma_{[\mathbf{ab}]}\right) && \text{(A7.3.5)}\\
&= \frac{1}{2} \omega_{\mathbf{a}} v^{[\mathbf{ab}]} \sigma_{[\mathbf{bc}]} + \frac{1}{2} \omega_{\mathbf{b}} v^{[\mathbf{ab}]} \sigma_{[\mathbf{ca}]} + \frac{1}{2!} v^{[\mathbf{ab}]} \sigma_{[\mathbf{ab}]} \omega_{\mathbf{c}} && \text{(A7.3.6)}\\
&= \frac{1}{2} \omega_{\mathbf{a}} v^{[\mathbf{ab}]} \sigma_{[\mathbf{bc}]} + \frac{1}{2} \omega_{\mathbf{b}} v^{[\mathbf{ba}]} \sigma_{[\mathbf{ac}]} + \frac{1}{2!} v^{[\mathbf{ab}]} \sigma_{[\mathbf{ab}]} \omega_{\mathbf{c}} && \text{(A7.3.7)}\\
&= \omega_{\mathbf{a}} v^{[\mathbf{ab}]} \sigma_{[\mathbf{bc}]} + \frac{1}{2!} v^{[\mathbf{ab}]} \sigma_{[\mathbf{ab}]} \omega_{\mathbf{c}} && \text{(A7.3.8)}\\
&\leftrightarrow \left(\underline{\omega} \cdot \vec{\vec{v}}\right) \cdot \underline{\underline{\sigma}} + \left(\vec{\vec{v}} \cdot \underline{\underline{\sigma}}\right) \underline{\omega} && \text{(A7.3.9)}
\end{aligned}$$