Homework 8 - Metric

Q8.1. Let $\vec{e}_r$ and $\vec{e}_\theta$ be the coordinate basis vectors associated with polar coordinates in two dimensional Euclidean space, and $\vec{e}_{\hat{r}}$ and $\vec{e}_{\hat{\theta}}$ be the orthonormal basis vectors proportional to $\vec{e}_r$ and $\vec{e}_\theta$.

(a) Express the metric $g_{\mathbf{ab}}$ and inverse metric $g^{\mathbf{ab}}$ in terms of the coordinate and orthonormal bases.

(b) Express the coordinate basis vectors and covectors in terms of the orthonormal basis vectors and covectors.

(c) Draw simple diagrams illustrating $\vec{e}_r$, $\vec{e}_\theta$, $\underline{e}^r$, $\underline{e}^\theta$ $\vec{e}_{\hat{r}}$, $\vec{e}_{\hat{\theta}}$, $\underline{e}^{\hat{r}}$ and $\underline{e}^{\hat{\theta}}$.

(d) Express the volume form $\epsilon_{\mathbf{ab}}$ and volume element $\epsilon^{\mathbf{ab}}$ in terms of the coordinate and orthonormal bases. Draw simple diagrams illustrating the origin of the extra factors in the case of the coordinate basis.

Q8.2. What is the geometrical meaning of $\left(\vec{u} \cdot \vec{\vec{v}}\right) \cdot \left(\vec{u} \cdot \vec{\vec{v}}\right)$?

Q8.3. Express the traditional vector calculus curl of a vector field in terms of the exterior derivative, and hence show that

$$\vec{\nabla} \times \vec{v} = \frac{\vec{e}_\alpha}{\epsilon_{\alpha\beta\gamma}} \frac{\partial}{\partial x^\beta} \left(g_{\gamma\delta} v^\delta\right) \qquad \text{(Q8.3.1)}$$

Hence derive the textbook formula for $\vec{\nabla} \times \vec{v}$ in spherical polar coordinates. Compare with the formula for $\underline{\nabla} \wedge \underline{v}$ in spherical polar coordinates.