Homework 8 - Metric

Q8.1. Let $\vec{e}_r$ and $\vec{e}_\theta$ be the coordinate basis vectors associated with polar coordinates in two dimensional Euclidean space, and $\vec{e}_{\hat{r}}$ and $\vec{e}_{\hat{\theta}}$ be the orthonormal basis vectors proportional to $\vec{e}_r$ and $\vec{e}_\theta$.

(a) Express the metric $g_{\mathbf{ab}}$ and inverse metric $g^{\mathbf{ab}}$ in terms of the coordinate and orthonormal bases.

(b) Express the coordinate basis vectors and covectors in terms of the orthonormal basis vectors and covectors.

(c) Draw simple diagrams illustrating $\vec{e}_r$, $\vec{e}_\theta$, $\underline{e}^r$, $\underline{e}^\theta$ $\vec{e}_{\hat{r}}$, $\vec{e}_{\hat{\theta}}$, $\underline{e}^{\hat{r}}$ and $\underline{e}^{\hat{\theta}}$.

(d) Express the volume form $\epsilon_{\mathbf{ab}}$ and volume element $\epsilon^{\mathbf{ab}}$ in terms of the coordinate and orthonormal bases. Draw simple diagrams illustrating the origin of the extra factors in the case of the coordinate basis.

A8.1. (a) In polar coordinates

$$g_{\alpha\beta}dx^\alpha dx^\beta = ds^2 = dr^2 + r^2 d\theta^2 \quad \text{(A8.1.1)}$$

therefore $g_{rr} = 1$, $g_{r\theta} = 0$ and $g_{\theta\theta} = r^2$, and using $g_{\alpha\beta}g^{\beta\gamma} = \delta_\alpha^\gamma$ gives $g^{rr} = 1$, $g^{r\theta} = 0$ and $g^{\theta\theta} = r^{-2}$. Therefore

$$g_{\mathbf{ab}} = g_{\alpha\beta}e^\alpha_{\mathbf{a}}e^\beta_{\mathbf{b}} = e^r_{\mathbf{a}}e^r_{\mathbf{b}} + r^2 e^\theta_{\mathbf{a}}e^\theta_{\mathbf{b}} \quad \text{(A8.1.2)}$$

and

$$g^{\mathbf{ab}} = g^{\alpha\beta}e_\alpha^{\mathbf{a}}e_\beta^{\mathbf{b}} = e_r^{\mathbf{a}}e_r^{\mathbf{b}} + \frac{1}{r^2}e_\theta^{\mathbf{a}}e_\theta^{\mathbf{b}} \quad \text{(A8.1.3)}$$

In an orthonormal basis

$$g_{\alpha\beta} = g_{\mathbf{ab}}e_\alpha^{\mathbf{a}}e_\beta^{\mathbf{b}} = \delta_{\alpha\beta} \quad \text{(A8.1.4)}$$

therefore

$$g_{\mathbf{ab}} = e^{\hat{r}}_{\mathbf{a}}e^{\hat{r}}_{\mathbf{b}} + e^{\hat{\theta}}_{\mathbf{a}}e^{\hat{\theta}}_{\mathbf{b}} \quad \text{(A8.1.5)}$$

and

$$g^{\mathbf{ab}} = e_{\hat{r}}^{\mathbf{a}}e_{\hat{r}}^{\mathbf{b}} + e_{\hat{\theta}}^{\mathbf{a}}e_{\hat{\theta}}^{\mathbf{b}} \quad \text{(A8.1.6)}$$

(b) Comparing Eqs. (A8.1.2) and (A8.1.5) and Eqs. (A8.1.3) and (A8.1.6) gives

$$\underline{e}^r = \underline{e}^{\hat{r}} \quad , \quad \underline{e}^\theta = \frac{1}{r}\underline{e}^{\hat{\theta}} \quad \text{(A8.1.7)}$$

$$\vec{e}_r = \vec{e}_{\hat{r}} \quad , \quad \vec{e}_\theta = r\vec{e}_{\hat{\theta}} \quad \text{(A8.1.8)}$$

(c) See Figure A8.1.1.

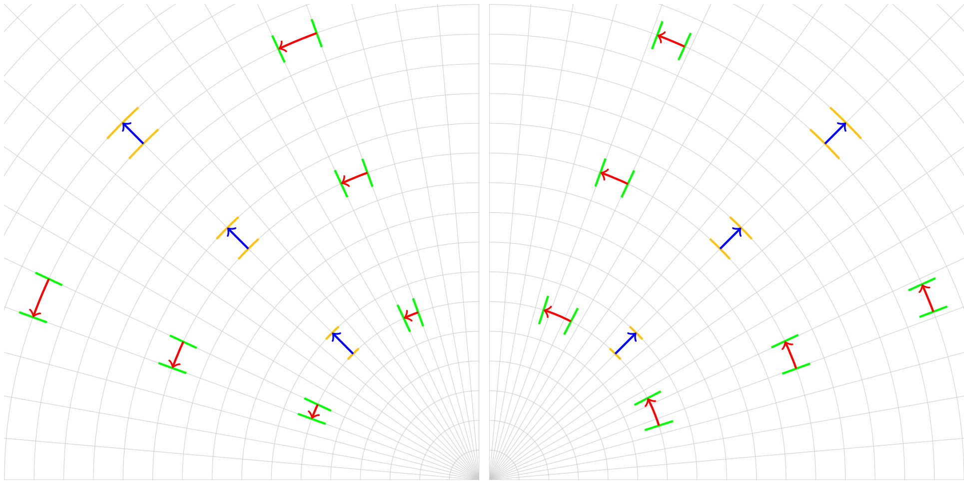

Figure A8.1.1: Left: polar coordinate basis, $\vec{e}_r$, $\vec{e}_\theta$, $\underline{e}^r$, $\underline{e}^\theta$. Right: polar orthonormal basis, $\vec{e}_{\hat{r}}$, $\vec{e}_{\hat{\theta}}$, $\underline{e}^{\hat{r}}$, $\underline{e}^{\hat{\theta}}$. Comparing: $\vec{e}_r = \vec{e}_{\hat{r}}$, $\underline{e}^r = \underline{e}^{\hat{r}}$, $\vec{e}_\theta = r\vec{e}_{\hat{\theta}}$, $\underline{e}^\theta = r^{-1}\underline{e}^{\hat{\theta}}$.

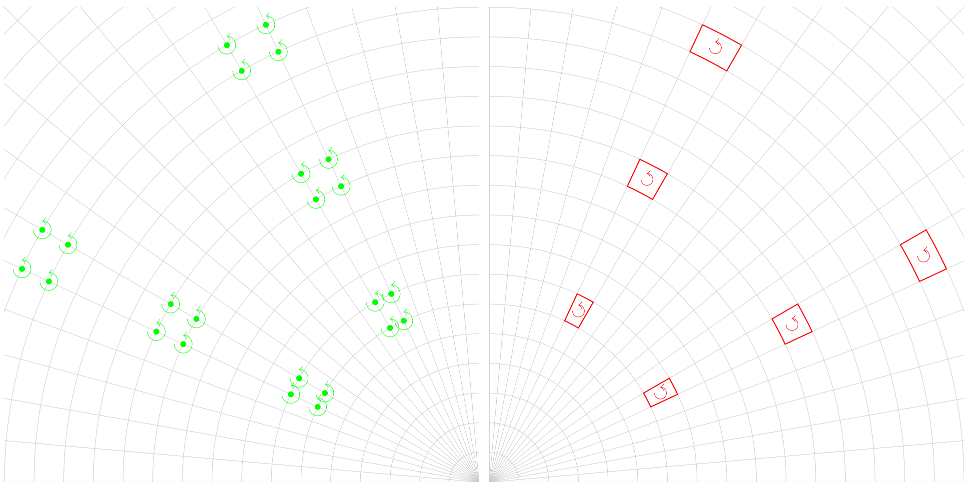

Figure A8.1.2: Polar coordinate volume form $\underline{e}^r \wedge \underline{e}^\theta = r^{-1}\underline{\underline{\epsilon}}$ and element $\vec{e}_r \wedge \vec{e}_\theta = r\vec{\vec{\epsilon}}$.

(d)

$$\underline{\underline{\epsilon}} = \epsilon_{12}\,\underline{e}^1 \wedge \underline{e}^2 = r\,\underline{e}^r \wedge \underline{e}^\theta \tag{A8.1.9}$$

$$= \underline{e}^{\hat{r}} \wedge \underline{e}^{\hat{\theta}} \tag{A8.1.10}$$

The polar coordinate volume form is inversely proportional to r, see Figure A8.1.2.

$$\vec{\vec{\epsilon}} = \frac{1}{\epsilon_{12}}\,\vec{e}_1 \wedge \vec{e}_2 = \frac{1}{r}\,\vec{e}_r \wedge \vec{e}_\theta \tag{A8.1.11}$$

$$= \vec{e}_{\hat{r}} \wedge \vec{e}_{\hat{\theta}} \tag{A8.1.12}$$

The polar coordinate volume element is proportional to r, see Figure A8.1.2.

Q8.2. What is the geometrical meaning of $\left(\vec{u}\cdot\vec{\vec{v}}\right)\cdot\left(\vec{u}\cdot\vec{\vec{v}}\right)$?

A8.2. Decomposing[1]

$$\vec{\vec{v}} = \vec{s} \wedge \vec{t} \tag{A8.2.1}$$

with

$$\vec{t} = \vec{u}\cdot\vec{\vec{v}} \tag{A8.2.2}$$

$$\vec{s} = \frac{\vec{\vec{v}}\cdot\vec{t}}{\vec{t}\cdot\vec{t}} \tag{A8.2.3}$$

so that

$$\vec{t}\cdot\vec{u} = 0 \tag{A8.2.4}$$

$$\vec{s}\cdot\vec{t} = 0 \tag{A8.2.5}$$

$$\vec{s}\cdot\vec{u} = 1 \tag{A8.2.6}$$

gives

$$\left(\vec{u}\cdot\vec{\vec{v}}\right)\cdot\left(\vec{u}\cdot\vec{\vec{v}}\right) = \left(\vec{u}\cdot\vec{s}\right)^2\left(\vec{t}\cdot\vec{t}\right) \tag{A8.2.7}$$

$$= |\vec{u}|^2\,|\vec{s}|^2\,\left|\vec{t}\right|^2\cos^2\theta \tag{A8.2.8}$$

$$= |\vec{u}|^2\left|\vec{\vec{v}}\right|^2\cos^2\theta \tag{A8.2.9}$$

where θ is the angle between $\vec{u}$ and $\vec{\vec{v}}$, see Figure A8.2.1.

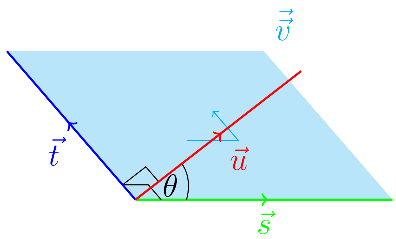

Figure A8.2.1: $\vec{u}\cdot\vec{\vec{v}} = \vec{u}\cdot\left(\vec{s}\wedge\vec{t}\right) = \left(\vec{u}\cdot\vec{s}\right)\vec{t} = \left(|\vec{u}|\,|\vec{s}|\cos\theta\right)\vec{t}$.

Q8.3. Express the traditional vector calculus curl of a vector field in terms of the exterior derivative, and hence show that

$$\vec{\nabla}\times\vec{v} = \frac{\vec{e}_\alpha}{\epsilon_{\alpha\beta\gamma}}\frac{\partial}{\partial x^\beta}\left(g_{\gamma\delta}v^\delta\right) \tag{Q8.3.1}$$

Hence derive the textbook formula for $\vec{\nabla}\times\vec{v}$ in spherical polar coordinates. Compare with the formula for $\underline{\nabla}\wedge\underline{v}$ in spherical polar coordinates.

[1] In four or more dimensions, it may only be possible to decompose $\vec{\vec{v}}$ into a sum of such terms from independent planes.

A8.3. Using Eqs. (1.5.22), (1.4.4) and (1.5.15),

$$\begin{aligned}\vec{\nabla}\times\vec{v} &= \star\underline{\nabla}\wedge\underline{v} && \text{(A8.3.1)}\\ &= \star\underline{\nabla}\wedge\left(\underline{e}^{\gamma}g_{\gamma\delta}v^{\delta}\right) && \text{(A8.3.2)}\\ &= \star\left(\underline{e}^{\beta}\wedge\underline{e}^{\gamma}\right)\frac{\partial}{\partial x^{\beta}}\left(g_{\gamma\delta}v^{\delta}\right) && \text{(A8.3.3)}\\ &= \frac{\vec{e}_{\alpha}}{\epsilon_{\alpha\beta\gamma}}\frac{\partial}{\partial x^{\beta}}\left(g_{\gamma\delta}v^{\delta}\right) && \text{(A8.3.4)}\end{aligned}$$

In three dimensional polar coordinates, the above becomes

$$\begin{aligned}\vec{\nabla}\times\vec{v} &= \frac{1}{\epsilon_{r\theta\phi}}\left[\frac{\partial}{\partial\theta}\left(g_{\phi\phi}v^{\phi}\right)-\frac{\partial}{\partial\phi}\left(g_{\theta\theta}v^{\theta}\right)\right]\vec{e}_r\\ &\quad+\frac{1}{\epsilon_{r\theta\phi}}\left[\frac{\partial}{\partial\phi}\left(g_{rr}v^{r}\right)-\frac{\partial}{\partial r}\left(g_{\phi\phi}v^{\phi}\right)\right]\vec{e}_\theta\\ &\quad+\frac{1}{\epsilon_{r\theta\phi}}\left[\frac{\partial}{\partial r}\left(g_{\theta\theta}v^{\theta}\right)-\frac{\partial}{\partial\theta}\left(g_{rr}v^{r}\right)\right]\vec{e}_\phi && \text{(A8.3.5)}\end{aligned}$$

Now $g_{rr}=1$, $g_{\theta\theta}=r^2$, $g_{\phi\phi}=r^2\sin^2\theta$ and $\epsilon_{r\theta\phi}=r^2\sin\theta$, therefore

$$\begin{aligned}\vec{\nabla}\times\vec{v} &= \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin^2\theta\, v^{\phi}\right)-\frac{\partial v^{\theta}}{\partial\phi}\right]\vec{e}_r\\ &\quad+\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial v^{r}}{\partial\phi}-\frac{\partial}{\partial r}\left(r^2\sin\theta\, v^{\phi}\right)\right]\vec{e}_\theta\\ &\quad+\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 v^{\theta}\right)-\frac{\partial v^{r}}{\partial\theta}\right]\vec{e}_\phi && \text{(A8.3.6)}\end{aligned}$$

The textbook formula uses the orthonormal basis

$$\vec{e}_{\hat{r}}=\vec{e}_r \quad,\quad \vec{e}_{\hat{\theta}}=\frac{1}{r}\vec{e}_\theta \quad,\quad \vec{e}_{\hat{\phi}}=\frac{1}{r\sin\theta}\vec{e}_\phi \qquad \text{(A8.3.7)}$$

with components

$$v^{\hat{r}}=v^{r} \quad,\quad v^{\hat{\theta}}=rv^{\theta} \quad,\quad v^{\hat{\phi}}=r\sin\theta\, v^{\phi} \qquad \text{(A8.3.8)}$$

Therefore the textbook formula is

$$\begin{aligned}\vec{\nabla}\times\vec{v} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\, v^{\hat{\phi}}\right)-\frac{\partial v^{\hat{\theta}}}{\partial\phi}\right]\vec{e}_{\hat{r}}\\ &\quad+\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v^{\hat{r}}}{\partial\phi}-\frac{\partial}{\partial r}\left(rv^{\hat{\phi}}\right)\right]\vec{e}_{\hat{\theta}}\\ &\quad+\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rv^{\hat{\theta}}\right)-\frac{\partial v^{\hat{r}}}{\partial\theta}\right]\vec{e}_{\hat{\phi}} && \text{(A8.3.9)}\end{aligned}$$

In comparison

$$\underline{\nabla}\wedge\underline{v}=\left(\frac{\partial v_\phi}{\partial\theta}-\frac{\partial v_\theta}{\partial\phi}\right)\underline{e}^{\theta}\wedge\underline{e}^{\phi}+\left(\frac{\partial v_r}{\partial\phi}-\frac{\partial v_\phi}{\partial r}\right)\underline{e}^{\phi}\wedge\underline{e}^{r}+\left(\frac{\partial v_\theta}{\partial r}-\frac{\partial v_r}{\partial\theta}\right)\underline{e}^{r}\wedge\underline{e}^{\theta} \qquad \text{(A8.3.10)}$$