Homework 10 - Curvature

Q10.1. Calculate

$$(\nabla_{\mathbf{a}}\nabla_{\mathbf{b}} - \nabla_{\mathbf{b}}\nabla_{\mathbf{a}})\, v^{\mathbf{c}} \tag{Q10.1.1}$$

in two different ways, and hence show that

$$R_{\mathbf{abcd}} = -R_{\mathbf{abdc}} \tag{Q10.1.2}$$

Find all possible distinct contractions of the metric with the curvature tensor.

Q10.2. Use the curvature tensor identity

$$R_{\mathbf{abc}}{}^{\mathbf{d}} + R_{\mathbf{bca}}{}^{\mathbf{d}} + R_{\mathbf{cab}}{}^{\mathbf{d}} = 0 \tag{Q10.2.1}$$

to show that

$$R_{\mathbf{abcd}} = R_{\mathbf{cdab}} \tag{Q10.2.2}$$

Q10.3. The operator $\triangle$ acting on an n-form $\boldsymbol{\omega}$ is defined by

$$\triangle\boldsymbol{\omega} = -\underline{\nabla} \wedge (\underline{\nabla} \cdot \boldsymbol{\omega}) - \underline{\nabla} \cdot (\underline{\nabla} \wedge \boldsymbol{\omega}) \tag{Q10.3.1}$$

(a) Express $\triangle\omega$ in terms of

$$\nabla^2 = g^{\mathbf{ab}}\nabla_{\mathbf{a}}\nabla_{\mathbf{b}} \tag{Q10.3.2}$$

(b) Show that

$$\triangle\omega = -\frac{1}{\epsilon_{1\cdots N}}\frac{\partial}{\partial x^\alpha}\left(\epsilon_{1\cdots N}\, g^{\alpha\beta}\frac{\partial\omega}{\partial x^\beta}\right) \tag{Q10.3.3}$$

(c) Express $\triangle\underline{\omega}$ in terms of

$$\nabla^2 = g^{\mathbf{ab}}\nabla_{\mathbf{a}}\nabla_{\mathbf{b}} \tag{Q10.3.4}$$

(d) Using

$$\underline{\underline{G}} = *\underline{\underline{F}} \tag{Q10.3.5}$$

show that

$$\triangle\underline{\underline{F}} = \underline{\nabla} \wedge *^{-1}\underline{\underline{\underline{J}}} \tag{Q10.3.6}$$

(e) In Lorentz gauge

$$\underline{\nabla} \cdot \underline{A} = 0 \tag{Q10.3.7}$$

show that

$$\triangle\underline{A} = *^{-1}\underline{\underline{\underline{J}}} \tag{Q10.3.8}$$