Homework 10 - Curvature

Q10.1. Calculate

$$\left(\nabla_{\mathbf{a}}\nabla_{\mathbf{b}} - \nabla_{\mathbf{b}}\nabla_{\mathbf{a}}\right) v^{\mathbf{c}} \tag{Q10.1.1}$$

in two different ways, and hence show that

$$R_{\mathbf{abcd}} = -R_{\mathbf{abdc}} \tag{Q10.1.2}$$

Find all possible distinct contractions of the metric with the curvature tensor.

A10.1. Using Eq. (2.2.4),

$$\left(\nabla_{\mathbf{a}}\nabla_{\mathbf{b}} - \nabla_{\mathbf{b}}\nabla_{\mathbf{a}}\right) v^{\mathbf{c}} = g^{\mathbf{ce}}\left(\nabla_{\mathbf{a}}\nabla_{\mathbf{b}} - \nabla_{\mathbf{b}}\nabla_{\mathbf{a}}\right) v_{\mathbf{e}} = g^{\mathbf{ce}} R_{\mathbf{abe}}{}^{\mathbf{d}} v_{\mathbf{d}} = g^{\mathbf{ce}} R_{\mathbf{abed}} v^{\mathbf{d}} \tag{A10.1.1}$$

and

$$\begin{aligned} 0 &= \left(\nabla_{\mathbf{a}}\nabla_{\mathbf{b}} - \nabla_{\mathbf{b}}\nabla_{\mathbf{a}}\right)\left(v^{\mathbf{d}}\omega_{\mathbf{d}}\right) && \text{(A10.1.2)} \\ &= \omega_{\mathbf{d}}\left(\nabla_{\mathbf{a}}\nabla_{\mathbf{b}} - \nabla_{\mathbf{b}}\nabla_{\mathbf{a}}\right) v^{\mathbf{d}} + v^{\mathbf{d}}\left(\nabla_{\mathbf{a}}\nabla_{\mathbf{b}} - \nabla_{\mathbf{b}}\nabla_{\mathbf{a}}\right)\omega_{\mathbf{d}} && \text{(A10.1.3)} \\ &= \omega_{\mathbf{c}}\left(\nabla_{\mathbf{a}}\nabla_{\mathbf{b}} - \nabla_{\mathbf{b}}\nabla_{\mathbf{a}}\right) v^{\mathbf{c}} + v^{\mathbf{d}} R_{\mathbf{abd}}{}^{\mathbf{c}}\omega_{\mathbf{c}} && \text{(A10.1.4)} \end{aligned}$$

therefore

$$\left(\nabla_{\mathbf{a}}\nabla_{\mathbf{b}} - \nabla_{\mathbf{b}}\nabla_{\mathbf{a}}\right) v^{\mathbf{c}} = -R_{\mathbf{abd}}{}^{\mathbf{c}} v^{\mathbf{d}} = -g^{\mathbf{ce}} R_{\mathbf{abde}} v^{\mathbf{d}} \tag{A10.1.5}$$

Comparing Eqs. (A10.1.1) and (A10.1.5) gives Eq. (Q10.1.2).
Eq. (2.2.4) implies

$$R_{\mathbf{abcd}} = -R_{\mathbf{bacd}} \tag{A10.1.6}$$

and Eqs. (A10.1.6) and (Q10.1.2) imply

$$g^{\mathbf{cd}} R_{\mathbf{cdab}} = g^{\mathbf{cd}} R_{\mathbf{abcd}} = 0 \tag{A10.1.7}$$

and

$$g^{\mathbf{cd}} R_{\mathbf{acbd}} = -g^{\mathbf{cd}} R_{\mathbf{acdb}} = -g^{\mathbf{cd}} R_{\mathbf{cabd}} = g^{\mathbf{cd}} R_{\mathbf{cadb}} \equiv R_{\mathbf{ab}} \tag{A10.1.8}$$

which is called the **Ricci tensor**. Further

$$g^{\mathbf{ab}} R_{\mathbf{ab}} \equiv R \tag{A10.1.9}$$

which is called the **Ricci scalar**.

Q10.2. Use the curvature tensor identity

$$R_{\mathbf{abc}}{}^{\mathbf{d}} + R_{\mathbf{bca}}{}^{\mathbf{d}} + R_{\mathbf{cab}}{}^{\mathbf{d}} = 0 \tag{Q10.2.1}$$

to show that

$$R_{\mathbf{abcd}} = R_{\mathbf{cdab}} \tag{Q10.2.2}$$

A10.2. Using Eqs. (Q10.2.1), (Q10.1.2) and (A10.1.6),

$$
\begin{aligned}
0 &= g_{\mathbf{de}}\left(R_{\mathbf{abc}}{}^{\mathbf{e}} + R_{\mathbf{bca}}{}^{\mathbf{e}} + R_{\mathbf{cab}}{}^{\mathbf{e}}\right) && \text{(A10.2.1)} \\
&= R_{\mathbf{abcd}} + R_{\mathbf{bcad}} + R_{\mathbf{cabd}} && \text{(A10.2.2)} \\
&= R_{\mathbf{abcd}} - R_{\mathbf{bcda}} - R_{\mathbf{cadb}} && \text{(A10.2.3)} \\
&= R_{\mathbf{abcd}} + R_{\mathbf{cdba}} + R_{\mathbf{dbca}} + R_{\mathbf{adcb}} + R_{\mathbf{dcab}} && \text{(A10.2.4)} \\
&= R_{\mathbf{abcd}} - R_{\mathbf{cdab}} - R_{\mathbf{dbac}} - R_{\mathbf{adbc}} - R_{\mathbf{cdab}} && \text{(A10.2.5)} \\
&= R_{\mathbf{abcd}} - R_{\mathbf{cdab}} + R_{\mathbf{badc}} - R_{\mathbf{cdab}} && \text{(A10.2.6)} \\
&= 2\left(R_{\mathbf{abcd}} - R_{\mathbf{cdab}}\right) && \text{(A10.2.7)}
\end{aligned}
$$

Q10.3. The operator $\triangle$ acting on an n-form $\boldsymbol{\omega}$ is defined by

$$\triangle\boldsymbol{\omega} = -\underline{\nabla} \wedge (\underline{\nabla} \cdot \boldsymbol{\omega}) - \underline{\nabla} \cdot (\underline{\nabla} \wedge \boldsymbol{\omega}) \tag{Q10.3.1}$$

(a) Express $\triangle\omega$ in terms of

$$\nabla^2 = g^{\mathbf{ab}} \nabla_{\mathbf{a}} \nabla_{\mathbf{b}} \tag{Q10.3.2}$$

(b) Show that

$$\triangle\omega = -\frac{1}{\epsilon_{1\cdots N}} \frac{\partial}{\partial x^\alpha} \left(\epsilon_{1\cdots N}\, g^{\alpha\beta} \frac{\partial \omega}{\partial x^\beta} \right) \tag{Q10.3.3}$$

(c) Express $\triangle\underline{\omega}$ in terms of

$$\nabla^2 = g^{\mathbf{ab}} \nabla_{\mathbf{a}} \nabla_{\mathbf{b}} \tag{Q10.3.4}$$

(d) Using

$$\underline{\underline{G}} = *\underline{\underline{F}} \tag{Q10.3.5}$$

show that

$$\triangle\underline{\underline{F}} = \underline{\nabla} \wedge *^{-1} \underline{\underline{\underline{J}}} \tag{Q10.3.6}$$

(e) In Lorentz gauge

$$\underline{\nabla} \cdot \underline{A} = 0 \tag{Q10.3.7}$$

show that

$$\triangle\underline{A} = *^{-1} \underline{\underline{\underline{J}}} \tag{Q10.3.8}$$

A10.3. (a)

$$\underline{\nabla} \cdot \omega = 0 \tag{A10.3.1}$$

and, using Eqs. (2.1.12), (2.1.16) and (2.1.57),

$$\underline{\nabla} \cdot (\underline{\nabla} \wedge \omega) = \nabla_{\mathbf{b}} g^{\mathbf{ba}} \nabla_{\mathbf{a}} \omega = \nabla^2 \omega \tag{A10.3.2}$$

therefore

$$\triangle\omega = -\nabla^2 \omega \tag{A10.3.3}$$

(b)

$$\underline{\nabla} \cdot \omega = 0 \tag{A10.3.4}$$

while Eq. (1.5.22) gives

$$\underline{\nabla} \wedge \omega = \frac{\partial \omega}{\partial x^\beta} \underline{e}^\beta \tag{A10.3.5}$$

using

$$g^{\mathbf{ab}} e^\beta_{\mathbf{b}} = g^{\alpha\beta} e^{\mathbf{a}}_\alpha \tag{A10.3.6}$$

gives

$$\diamond \underline{\nabla} \wedge \omega = g^{\alpha\beta} \frac{\partial \omega}{\partial x^\beta} \vec{e}_\alpha \tag{A10.3.7}$$

and using Eq. (Q6.2.1) gives

$$\underline{\nabla} \cdot \diamond \underline{\nabla} \wedge \omega = \frac{1}{\epsilon_{1\cdots N}} \frac{\partial}{\partial x^\alpha} \left(\epsilon_{1\cdots N} g^{\alpha\beta} \frac{\partial \omega}{\partial x^\beta} \right) \tag{A10.3.8}$$

Substituting Eqs. (A10.3.4) and (A10.3.8) into Eq. (Q10.3.1) gives Eq. (Q10.3.3).

(c) Using Eqs. (2.1.12), (2.1.17), (2.1.57), (2.2.4) and (Q10.3.4),

$$\begin{aligned}
\triangle \omega_{\mathbf{a}} &= -\left[\underline{\nabla} \cdot (\underline{\nabla} \wedge \underline{\omega})\right]_{\mathbf{a}} - \left[\underline{\nabla} \wedge (\underline{\nabla} \cdot \underline{\omega})\right]_{\mathbf{a}} && \text{(A10.3.9)} \\
&= -\nabla_{\mathbf{c}} g^{\mathbf{cb}} \left(\nabla_{\mathbf{b}} \omega_{\mathbf{a}} - \nabla_{\mathbf{a}} \omega_{\mathbf{b}}\right) - \nabla_{\mathbf{a}} \nabla_{\mathbf{c}} g^{\mathbf{cb}} \omega_{\mathbf{b}} && \text{(A10.3.10)} \\
&= -g^{\mathbf{cb}} \nabla_{\mathbf{c}} \nabla_{\mathbf{b}} \omega_{\mathbf{a}} + g^{\mathbf{cb}} \left(\nabla_{\mathbf{c}} \nabla_{\mathbf{a}} - \nabla_{\mathbf{a}} \nabla_{\mathbf{c}}\right) \omega_{\mathbf{b}} && \text{(A10.3.11)} \\
&= -\nabla^2 \omega_{\mathbf{a}} + g^{\mathbf{cb}} R_{\mathbf{cab}}{}^{\mathbf{d}} \omega_{\mathbf{d}} && \text{(A10.3.12)} \\
&= -\nabla^2 \omega_{\mathbf{a}} + R_{\mathbf{a}}{}^{\mathbf{b}} \omega_{\mathbf{b}} && \text{(A10.3.13)}
\end{aligned}$$

(d) Eq. (1.3.48) is

$$\underline{\nabla} \wedge \underline{\underline{G}} = \underline{\underline{\underline{J}}} \tag{A10.3.14}$$

and using Eqs. (Q10.3.5) and (2.1.56) gives

$$-\underline{\nabla} \cdot \underline{\underline{F}} = *^{-1} \underline{\underline{\underline{J}}} \tag{A10.3.15}$$

Substituting Eqs. (A10.3.15) and (1.3.47) into Eq. (Q10.3.1) gives Eq. (Q10.3.6).

(e) Eqs. (A10.3.15) and (1.3.49) give

$$-\underline{\nabla} \cdot (\underline{\nabla} \wedge \underline{A}) = *^{-1} \underline{\underline{\underline{J}}} \tag{A10.3.16}$$

Substituting Eqs. (Q10.3.7) and (A10.3.16) into Eq. (Q10.3.1) gives Eq. (Q10.3.8).