Homework 11 - Christoffel symbols

Q11.1. Show that

$$\nabla_{\mathbf{a}} e^{\gamma}_{\mathbf{b}} = -\Gamma^{\gamma}_{\alpha\beta} e^{\alpha}_{\mathbf{a}} e^{\beta}_{\mathbf{b}} \tag{Q11.1.1}$$

and hence derive Eq. (2.2.9) and show that, for zero torsion and in a coordinate basis,

$$\Gamma^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\beta\alpha} \tag{Q11.1.2}$$

A11.1.

$$0 = \nabla_{\mathbf{a}} \delta^{\gamma}_{\beta} = \nabla_{\mathbf{a}} \left(e^{\mathbf{c}}_{\beta} e^{\gamma}_{\mathbf{c}} \right) = \left(\nabla_{\mathbf{a}} e^{\mathbf{c}}_{\beta} \right) e^{\gamma}_{\mathbf{c}} + e^{\mathbf{c}}_{\beta} \left(\nabla_{\mathbf{a}} e^{\gamma}_{\mathbf{c}} \right) \tag{A11.1.1}$$

therefore

$$\nabla_{\mathbf{a}} e^{\gamma}_{\mathbf{b}} = -e^{\beta}_{\mathbf{b}} \left(\nabla_{\mathbf{a}} e^{\mathbf{c}}_{\beta} \right) e^{\gamma}_{\mathbf{c}} = -e^{\beta}_{\mathbf{b}} \Gamma^{\mathbf{c}}_{\mathbf{a}\beta} e^{\gamma}_{\mathbf{c}} = -\Gamma^{\gamma}_{\alpha\beta} e^{\alpha}_{\mathbf{a}} e^{\beta}_{\mathbf{b}} \tag{A11.1.2}$$

and hence

$$\nabla_{\mathbf{a}} \omega_{\mathbf{b}} = \nabla_{\mathbf{a}} \left(\omega_{\beta} e^{\beta}_{\mathbf{b}} \right) = \left(\nabla_{\mathbf{a}} \omega_{\beta} \right) e^{\beta}_{\mathbf{b}} + \omega_{\gamma} \nabla_{\mathbf{a}} e^{\gamma}_{\mathbf{b}} = \left(\frac{\partial \omega_{\beta}}{\partial x^{\alpha}} - \omega_{\gamma} \Gamma^{\gamma}_{\alpha\beta} \right) e^{\alpha}_{\mathbf{a}} e^{\beta}_{\mathbf{b}} \tag{A11.1.3}$$

For zero torsion and in a coordinate basis,

$$\nabla_{\mathbf{a}} e^{\gamma}_{\mathbf{b}} = \nabla_{\mathbf{a}} \nabla_{\mathbf{b}} x^{\gamma} = \nabla_{\mathbf{b}} \nabla_{\mathbf{a}} x^{\gamma} = \nabla_{\mathbf{b}} e^{\gamma}_{\mathbf{a}} \tag{A11.1.4}$$

therefore from Eq. (A11.1.2)

$$\Gamma^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\beta\alpha} \tag{A11.1.5}$$

Q11.2. Derive Eq. (2.2.10).

A11.2. Eqs. (2.2.2) and (Q11.1.1) give

$$\begin{aligned} 0 &= \nabla_{\mathbf{a}} g_{\mathbf{bc}} && \text{(A11.2.1)} \\ &= \nabla_{\mathbf{a}} \left(g_{\beta\gamma} e^{\beta}_{\mathbf{b}} e^{\gamma}_{\mathbf{c}} \right) && \text{(A11.2.2)} \\ &= g_{\beta\gamma,\alpha} e^{\alpha}_{\mathbf{a}} e^{\beta}_{\mathbf{b}} e^{\gamma}_{\mathbf{c}} - g_{\beta\gamma} \Gamma^{\beta}_{\mathbf{ab}} e^{\gamma}_{\mathbf{c}} - g_{\beta\gamma} e^{\beta}_{\mathbf{b}} \Gamma^{\gamma}_{\mathbf{ac}} && \text{(A11.2.3)} \end{aligned}$$

where subscript $,\alpha$ denotes the partial derivative with respect to x^{α}. Therefore

$$g_{\beta\gamma,\alpha} = g_{\delta\gamma} \Gamma^{\delta}_{\alpha\beta} + g_{\beta\delta} \Gamma^{\delta}_{\alpha\gamma} \tag{A11.2.4}$$

and

$$\begin{aligned} & g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha} \\ &= g_{\delta\beta} \Gamma^{\delta}_{\gamma\alpha} + g_{\alpha\delta} \Gamma^{\delta}_{\gamma\beta} + g_{\delta\gamma} \Gamma^{\delta}_{\beta\alpha} + g_{\alpha\delta} \Gamma^{\delta}_{\beta\gamma} - g_{\delta\gamma} \Gamma^{\delta}_{\alpha\beta} - g_{\beta\delta} \Gamma^{\delta}_{\alpha\gamma} && \text{(A11.2.5)} \\ &= g_{\alpha\delta} \left(\Gamma^{\delta}_{\gamma\beta} + \Gamma^{\delta}_{\beta\gamma} \right) + g_{\delta\gamma} \left(\Gamma^{\delta}_{\beta\alpha} - \Gamma^{\delta}_{\alpha\beta} \right) + g_{\beta\delta} \left(\Gamma^{\delta}_{\gamma\alpha} - \Gamma^{\delta}_{\alpha\gamma} \right) && \text{(A11.2.6)} \end{aligned}$$

Therefore, using Eq. (Q11.1.2)

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(g_{\delta\beta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta} \right) \tag{A11.2.7}$$

Q11.3. Express the curvature tensor in terms of the Christoffel symbols.

A11.3. Using Eqs. (2.2.4) and (Q11.1.1),

$$R_{\mathbf{abc}}{}^{\mathbf{e}} e^{\delta}_{\mathbf{e}} = \left(\nabla_{\mathbf{a}}\nabla_{\mathbf{b}} - \nabla_{\mathbf{b}}\nabla_{\mathbf{a}}\right) e^{\delta}_{\mathbf{c}} = -\nabla_{\mathbf{a}}\Gamma^{\delta}_{\mathbf{bc}} + \nabla_{\mathbf{b}}\Gamma^{\delta}_{\mathbf{ac}} \tag{A11.3.1}$$

Therefore, using Eq. (2.2.7),

$$\begin{aligned}
& = -\nabla_{\mathbf{a}}\Gamma^{\mathbf{d}}_{\mathbf{bc}} + \Gamma^{\mathbf{d}}_{\mathbf{ae}}\Gamma^{\mathbf{e}}_{\mathbf{bc}} + \nabla_{\mathbf{b}}\Gamma^{\mathbf{d}}_{\mathbf{ac}} - \Gamma^{\mathbf{d}}_{\mathbf{be}}\Gamma^{\mathbf{e}}_{\mathbf{ac}} && \text{(A11.3.2)} \\
& = -\nabla_{\mathbf{a}}\Gamma^{\mathbf{d}}_{\mathbf{bc}} + \Gamma^{\delta}_{\mathbf{bc}}\nabla_{\mathbf{a}}e^{\mathbf{d}}_{\delta} + \nabla_{\mathbf{b}}\Gamma^{\mathbf{d}}_{\mathbf{ac}} - \Gamma^{\delta}_{\mathbf{ac}}\nabla_{\mathbf{b}}e^{\mathbf{d}}_{\delta} && \text{(A11.3.3)} \\
R_{\mathbf{abc}}{}^{\mathbf{d}} & = -e^{\mathbf{d}}_{\delta}\nabla_{\mathbf{a}}\Gamma^{\delta}_{\mathbf{bc}} + e^{\mathbf{d}}_{\delta}\nabla_{\mathbf{b}}\Gamma^{\delta}_{\mathbf{ac}} && \text{(A11.3.4)} \\
& = -e^{\mathbf{d}}_{\delta}\nabla_{\mathbf{a}}\left(\Gamma^{\delta}_{\beta\gamma}e^{\beta}_{\mathbf{b}}e^{\gamma}_{\mathbf{c}}\right) + e^{\mathbf{d}}_{\delta}\nabla_{\mathbf{b}}\left(\Gamma^{\delta}_{\alpha\gamma}e^{\alpha}_{\mathbf{a}}e^{\gamma}_{\mathbf{c}}\right) && \text{(A11.3.5)} \\
& = \left(-\nabla_{\alpha}\Gamma^{\delta}_{\beta\gamma} + \Gamma^{\delta}_{\epsilon\gamma}\Gamma^{\epsilon}_{\alpha\beta} + \Gamma^{\delta}_{\beta\epsilon}\Gamma^{\epsilon}_{\alpha\gamma} + \nabla_{\beta}\Gamma^{\delta}_{\alpha\gamma} - \Gamma^{\delta}_{\epsilon\gamma}\Gamma^{\epsilon}_{\beta\alpha} - \Gamma^{\delta}_{\alpha\epsilon}\Gamma^{\epsilon}_{\beta\gamma}\right) \\
& \quad \times e^{\alpha}_{\mathbf{a}}e^{\beta}_{\mathbf{b}}e^{\gamma}_{\mathbf{c}}e^{\mathbf{d}}_{\delta} && \text{(A11.3.6)} \\
& = \left(-\nabla_{\alpha}\Gamma^{\delta}_{\beta\gamma} - \Gamma^{\delta}_{\alpha\epsilon}\Gamma^{\epsilon}_{\beta\gamma} + \nabla_{\beta}\Gamma^{\delta}_{\alpha\gamma} + \Gamma^{\delta}_{\beta\epsilon}\Gamma^{\epsilon}_{\alpha\gamma}\right) e^{\alpha}_{\mathbf{a}}e^{\beta}_{\mathbf{b}}e^{\gamma}_{\mathbf{c}}e^{\mathbf{d}}_{\delta} && \text{(A11.3.7)}
\end{aligned}$$

where Eq. (A11.3.7) assumes a coordinate basis and uses Eq. (Q11.1.2).

Q11.4. Let $e^{\mathbf{a}}_{r}$ and $e^{\mathbf{a}}_{\theta}$ be the coordinate basis vectors associated with polar coordinates in two dimensional Euclidean space, and $e^{\mathbf{a}}_{\hat{r}}$ and $e^{\mathbf{a}}_{\hat{\theta}}$ be the orthonormal basis vectors proportional to $e^{\mathbf{a}}_{r}$ and $e^{\mathbf{a}}_{\theta}$.

(a) Calculate the $\Gamma^{\gamma}_{\alpha\beta}$ for the coordinate basis.

(b) Express the velocity and acceleration in terms of the coordinate and orthonormal bases.

(c) Write down the equation of a geodesic in the coordinate basis and check that the geodesics are straight lines.

A11.4. (a) Using Eqs. (A11.2.7) and (2.1.33), the only non-zero derivative of the metric components is

$$\frac{\partial g_{\theta\theta}}{\partial r} = 2r \tag{A11.4.1}$$

therefore

$$\Gamma^{r}_{\theta\theta} = -r \quad , \quad \Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{r} \tag{A11.4.2}$$

and others zero.

(b) Eqs. (2.2.11) and (A8.1.8) give

$$\begin{aligned}
v^{\mathbf{a}} = \frac{dx^{\alpha}}{dt}e^{\mathbf{a}}_{\alpha} & = \dot{r}e^{\mathbf{a}}_{r} + \dot{\theta}e^{\mathbf{a}}_{\theta} && \text{(A11.4.3)} \\
& = \dot{r}e^{\mathbf{a}}_{\hat{r}} + r\dot{\theta}e^{\mathbf{a}}_{\hat{\theta}} && \text{(A11.4.4)}
\end{aligned}$$

and Eqs. (2.2.12) and (A11.4.2) give

$$\begin{aligned} a^{\mathbf{a}} &= \left(\frac{d^2 x^\alpha}{dt^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{dt} \frac{dx^\gamma}{dt} \right) e^{\mathbf{a}}_\alpha && \text{(A11.4.5)} \\ &= \left(\ddot{r} - r\dot{\theta}^2 \right) e^{\mathbf{a}}_r + \left(\ddot{\theta} + \frac{2\dot{r}\dot{\theta}}{r} \right) e^{\mathbf{a}}_\theta && \text{(A11.4.6)} \\ &= \left(\ddot{r} - r\dot{\theta}^2 \right) e^{\mathbf{a}}_{\hat{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} \right) e^{\mathbf{a}}_{\hat{\theta}} && \text{(A11.4.7)} \end{aligned}$$

(c) The geodesic equation is

$$a^{\mathbf{a}} = 0 \qquad \text{(A11.4.8)}$$

therefore

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= 0 && \text{(A11.4.9)} \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} &= 0 && \text{(A11.4.10)} \end{aligned}$$

corresponding to straight lines, though not obviously so.