Homework 12 - Calculus of variations

Q12.1. The action functional

$$S[x(t)] = \int_{t_\mathrm{i}}^{t_\mathrm{f}} L(x, \dot{x}, t)\, dt \tag{Q12.1.1}$$

can be varied either covariantly

$$\frac{\delta S}{\delta x^{\mathbf{a}}} = \frac{\partial L}{\partial x^{\mathbf{a}}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^{\mathbf{a}}}\right) \tag{Q12.1.2}$$

or with respect to the coordinate paths

$$\frac{\delta S}{\delta x^{\alpha}} = \frac{\partial L}{\partial x^{\alpha}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^{\alpha}}\right) \tag{Q12.1.3}$$

(a) Show that

$$\frac{\delta S}{\delta x^{\mathbf{a}}} = \frac{\delta S}{\delta x^{\alpha}} e^{\alpha}_{\mathbf{a}} \tag{Q12.1.4}$$

(b) Evaluate Eqs. (Q12.1.2) and (Q12.1.3) for

$$L = \frac{1}{2} m g_{\mathbf{ab}} \dot{x}^{\mathbf{a}} \dot{x}^{\mathbf{b}} - V(x) \tag{Q12.1.5}$$

and show that they are equivalent.

A12.1. (a)

$$\begin{aligned}
\frac{\delta S}{\delta x^{\mathbf{a}}} &= \frac{\partial L}{\partial x^{\mathbf{a}}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^{\mathbf{a}}}\right) \\
&= \frac{\partial L}{\partial x^{\alpha}} e^{\alpha}_{\mathbf{a}} + \frac{\partial L}{\partial \dot{x}^{\alpha}} \dot{e}^{\alpha}_{\mathbf{a}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^{\alpha}} e^{\alpha}_{\mathbf{a}}\right) \\
&= \left[\frac{\partial L}{\partial x^{\alpha}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^{\alpha}}\right)\right] e^{\alpha}_{\mathbf{a}} \\
&= \frac{\delta S}{\delta x^{\alpha}} e^{\alpha}_{\mathbf{a}}
\end{aligned} \tag{A12.1.1}$$

(b)

$$L = \frac{1}{2} m g_{\mathbf{ab}} \dot{x}^{\mathbf{a}} \dot{x}^{\mathbf{b}} - V(x) \tag{A12.1.2}$$

therefore

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^{\mathbf{a}}}\right) - \frac{\partial L}{\partial x^{\mathbf{a}}} &= \frac{d}{dt}\left(m g_{\mathbf{ab}} \dot{x}^{\mathbf{b}}\right) + \frac{\partial V}{\partial x^{\mathbf{a}}} \\
&= m g_{\mathbf{ab}} \ddot{x}^{\mathbf{b}} + \nabla_{\mathbf{a}} V
\end{aligned} \tag{A12.1.3}$$

In components

$$L = \frac{1}{2} m g_{\gamma\delta} \dot{x}^{\gamma} \dot{x}^{\delta} - V(x) \tag{A12.1.4}$$

therefore

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) - \frac{\partial L}{\partial x^\alpha} &= \frac{d}{dt}\left(m g_{\alpha\beta}\dot{x}^\beta\right) - \frac{1}{2}m\frac{\partial g_{\gamma\delta}}{\partial x^\alpha}\dot{x}^\gamma\dot{x}^\delta + \frac{\partial V}{\partial x^\alpha} \\
&= m\dot{x}^\epsilon\frac{\partial g_{\alpha\beta}}{\partial x^\epsilon}\dot{x}^\beta + m g_{\alpha\beta}\ddot{x}^\beta - \frac{1}{2}m\frac{\partial g_{\gamma\delta}}{\partial x^\alpha}\dot{x}^\gamma\dot{x}^\delta + \frac{\partial V}{\partial x^\alpha} \\
&= m g_{\alpha\beta}\ddot{x}^\beta + \frac{1}{2}m\left(\frac{\partial g_{\alpha\gamma}}{\partial x^\delta} + \frac{\partial g_{\alpha\delta}}{\partial x^\gamma} - \frac{\partial g_{\gamma\delta}}{\partial x^\alpha}\right)\dot{x}^\gamma\dot{x}^\delta + \frac{\partial V}{\partial x^\alpha} \\
&= m g_{\alpha\beta}\left(\ddot{x}^\beta + \Gamma^\beta_{\gamma\delta}\dot{x}^\gamma\dot{x}^\delta\right) + \frac{\partial V}{\partial x^\alpha} \qquad \text{(A12.1.5)}
\end{aligned}$$

which is the component form of Eq. (A12.1.3).

Q12.2. The action functional for a scalar field can be expressed either geometrically

$$S[\phi(x)] = \int L(\phi, \nabla\phi, x)\,\boldsymbol{\epsilon} \qquad \text{(Q12.2.1)}$$

giving the geometric Euler-Lagrange equation

$$\nabla_{\mathbf{a}}\left[\frac{\partial L}{\partial(\nabla_{\mathbf{a}}\phi)}\right] - \frac{\partial L}{\partial\phi} = 0 \qquad \text{(Q12.2.2)}$$

or in terms of coordinates

$$S[\phi(x)] = \int L(\phi, \partial\phi, x)\,\sqrt{|g|}\,d^4x \qquad \text{(Q12.2.3)}$$

giving the coordinate Euler-Lagrange equation

$$\partial_\alpha\left[\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)}\right] - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \qquad \text{(Q12.2.4)}$$

where the Lagrangian density

$$\mathcal{L} = \sqrt{|g|}\,L \qquad \text{(Q12.2.5)}$$

(a) Show that Eqs. (Q12.2.2) and (Q12.2.4) are equivalent.

(b) Evaluate Eqs. (Q12.2.2) and (Q12.2.4) for

$$L = \frac{1}{2}g^{\mathbf{ab}}\left(\nabla_{\mathbf{a}}\phi\right)\left(\nabla_{\mathbf{b}}\phi\right) - V(\phi) \qquad \text{(Q12.2.6)}$$

and show that the resulting equations are equivalent.

A12.2. (a) Eqs. (Q6.2.1) and (2.1.50) and

$$\frac{\partial L}{\partial(\partial_\alpha\phi)} = e^\alpha_{\mathbf{a}}\frac{\partial L}{\partial(\nabla_{\mathbf{a}}\phi)} \qquad \text{(A12.2.1)}$$

give

$$\nabla_{\mathbf{a}}\left[\frac{\partial L}{\partial(\nabla_{\mathbf{a}}\phi)}\right]=\frac{1}{\sqrt{|g|}}\partial_\alpha\left[\sqrt{|g|}\,\frac{\partial L}{\partial(\partial_\alpha\phi)}\right] \tag{A12.2.2}$$

therefore

$$\nabla_{\mathbf{a}}\left[\frac{\partial L}{\partial(\nabla_{\mathbf{a}}\phi)}\right]-\frac{\partial L}{\partial\phi} \;=\; \frac{1}{\sqrt{|g|}}\left\{\partial_\alpha\left[\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)}\right]-\frac{\partial\mathcal{L}}{\partial\phi}\right\} \tag{A12.2.3}$$

(b)

$$L=\frac{1}{2}g^{\mathbf{ab}}\left(\nabla_{\mathbf{a}}\phi\right)\left(\nabla_{\mathbf{b}}\phi\right)-V(\phi) \tag{A12.2.4}$$

therefore

$$\nabla_{\mathbf{a}}\left[\frac{\partial L}{\partial(\nabla_{\mathbf{a}}\phi)}\right]-\frac{\partial L}{\partial\phi} \;=\; g^{\mathbf{ab}}\nabla_{\mathbf{a}}\nabla_{\mathbf{b}}\phi+\frac{\partial V}{\partial\phi} \tag{A12.2.5}$$

In components

$$\mathcal{L}=\sqrt{|g|}\left[\frac{1}{2}g^{\alpha\beta}\left(\partial_\alpha\phi\right)\left(\partial_\beta\phi\right)-V(\phi)\right] \tag{A12.2.6}$$

therefore, using Eqs. (Q10.3.2), (A10.3.3), (Q10.3.3) and (2.1.50),

$$\frac{1}{\sqrt{|g|}}\left\{\partial_\alpha\left[\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)}\right]-\frac{\partial\mathcal{L}}{\partial\phi}\right\} \;=\; \frac{1}{\sqrt{|g|}}\partial_\alpha\left(\sqrt{|g|}\,g^{\alpha\beta}\partial_\beta\phi\right)+\frac{\partial V}{\partial\phi} \tag{A12.2.7}$$

$$=\; g^{\mathbf{ab}}\nabla_{\mathbf{a}}\nabla_{\mathbf{b}}\phi+\frac{\partial V}{\partial\phi} \tag{A12.2.8}$$