1.2 Topological tensor calculus

1.2.1 Tensor fields

Finite displacements in Euclidean space can be represented by arrows and have a natural vector space structure, but finite displacements in more general curved spaces, such as on the surface of a sphere, do not. However, an infinitesimal neighborhood of a point in a smooth curved space[1] looks like an infinitesimal neighborhood of Euclidean space, and infinitesimal displacements $\vec{dx}$ retain the vector space structure of displacements in Euclidean space. An infinitesimal neighborhood of a point can be infinitely rescaled to generate a finite vector space, called the **tangent space**, at the point. A vector lives in the tangent space of a point. Note that vectors do not stretch from one point to another, and vectors at different points live in different tangent spaces and so cannot be added.

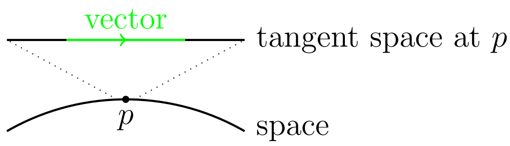

Figure 1.2.1: A vector in the tangent space of a point.

For example, rescaling the infinitesimal displacement $\vec{dx}$ by dividing it by the infinitesimal scalar dt gives the velocity

$$\vec{v} = \frac{\vec{dx}}{dt} \tag{1.2.1}$$

which is a vector. Similarly, we can picture the covector $\underline{\nabla}\phi$ as the infinitesimal contours of ϕ in a neighborhood of a point, infinitely rescaled to generate a finite covector in the point's cotangent space. More generally, infinitely rescaling the neighborhood of a point generates the **tensor space** and its algebra at the point. The tensor space contains the tangent and cotangent spaces as a vector subspaces.

A **tensor field** is something that takes tensor values at every point in a space. Tensor fields of the same type can be added, and multiplied by a scalar, in the usual way.

1.2.2 Exterior derivative

The **exterior derivative** [2]

$$\underline{\nabla} \wedge \boldsymbol{\omega} \tag{1.2.2}$$

of a differential form $\boldsymbol{\omega}$ is defined as the topological (antisymmetric) derivative so that the exterior derivative of a differential form is also a differential form. It does not

[1] In mathematical language, a smooth manifold.

[2] The mathematical notation for $\underline{\nabla} \wedge \boldsymbol{\omega}$ is $d\omega$.

depend on how the tensor spaces at different points are connected. This makes the exterior derivative $\underline{\nabla}\wedge$ simpler than the more general covariant derivative $\underline{\nabla}$ defined later, and gives it a clear physical interpretation.

For example, a scalar field can be thought of as a codimension zero plane density, and its exterior derivative is the one-form field given by the oriented edges of the scalar field's codimension zero planes, i.e. the contours of the scalar field. The exterior derivative of a one-form field is the two-form field given by the oriented edges of the one-form field's planes. See Figure 1.2.2. More generally, the exterior derivative of an n-form field is

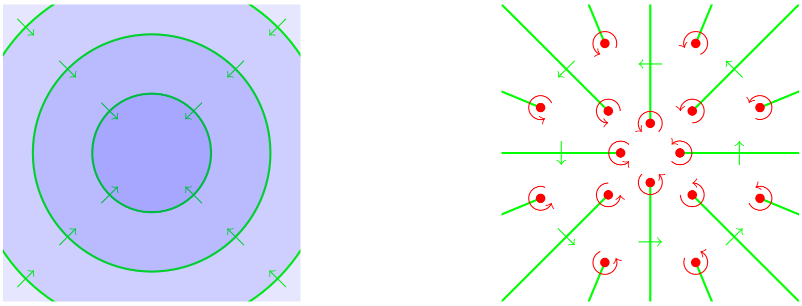

Figure 1.2.2: Left: $\underline{\nabla} \wedge \phi = \underline{\xi}$, right: $\underline{\nabla} \wedge \underline{\zeta} = \underline{\underline{\rho}}$.

the $(n+1)$-form field given by the oriented edges of the n-form field's codimension n planes. Thus the exterior derivative $\underline{\nabla}\wedge$ has the meaning 'the oriented boundaries of' and gives a measure of the spacial rate of change of the tensor field.

The exterior derivative has the defining properties:

- Acting on a scalar field, the exterior derivative is equal to the gradient

$$\underline{\nabla} \wedge \omega = \underline{\nabla} \omega \tag{1.2.3}$$

- For any differential form field $\boldsymbol{\omega}$,

$$\underline{\nabla} \wedge \underline{\nabla} \wedge \boldsymbol{\omega} = 0 \tag{1.2.4}$$

  since the boundary of a boundary is zero, as can be seen from Figure 1.2.2.

- Taking into account the antisymmetry of the wedge product, the Leibniz rule is

$$\underline{\nabla} \wedge (\boldsymbol{\omega} \wedge \boldsymbol{\sigma}) = (\underline{\nabla} \wedge \boldsymbol{\omega}) \wedge \boldsymbol{\sigma} + (-1)^{\deg \boldsymbol{\omega}} \boldsymbol{\omega} \wedge (\underline{\nabla} \wedge \boldsymbol{\sigma}) \tag{1.2.5}$$

The exterior derivative can be used to define the **Lie derivative** of a differential form field $\boldsymbol{\omega}$ with respect to a vector field $\vec{v}$ [3]

$$\mathcal{L}_v \boldsymbol{\omega} = \vec{v} \cdot (\underline{\nabla} \wedge \boldsymbol{\omega}) + \underline{\nabla} \wedge (\vec{v} \cdot \boldsymbol{\omega}) \tag{1.2.6}$$

The Lie derivative of a multivector field will be given in Section 2.3.3.

[3] This form for the Lie derivative is motivated by $(\vec{v} \cdot \underline{\delta})\, \boldsymbol{\omega} = \vec{v} \cdot (\underline{\delta} \wedge \boldsymbol{\omega}) + \underline{\delta} \wedge (\vec{v} \cdot \boldsymbol{\omega})$ for a one-form field $\underline{\delta}$.

1.2.3 Integration

An n-form field $\boldsymbol{\omega}$ naturally contracts with an n-dimensional surface S to give a scalar

$$\int_S \boldsymbol{\omega} = \text{scalar} \tag{1.2.7}$$

with the same interpretation as the contraction of an n-form with an n-vector. If we divide the surface S into infinitesimal surface elements $\boldsymbol{dS}$, the integral of $\boldsymbol{\omega}$ over S can be written in the more familiar form

$$\int_S \boldsymbol{\omega} \cdot \boldsymbol{dS} \tag{1.2.8}$$

For example, the integral of a current density $\underline{\underline{j}}$ over a surface S is the current I flowing through the surface

$$I = \int_S \underline{\underline{j}} \cdot d\vec{\vec{S}} \tag{1.2.9}$$

or the integral of a charge density $\underline{\underline{\underline{\rho}}}$ over a volume V is the charge Q contained in the volume

$$Q = \int_V \underline{\underline{\underline{\rho}}} \cdot d\vec{\vec{\vec{V}}} \tag{1.2.10}$$

Stokes' theorem states that

$$\int_S \underline{\nabla} \wedge \boldsymbol{\omega} = \int_{\partial S} \boldsymbol{\omega} \tag{1.2.11}$$

where ∂S is the boundary of S, see Figure 1.2.3.

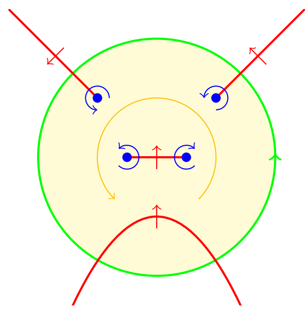

Figure 1.2.3: Stokes' theorem: $\int_S \underline{\nabla} \wedge \boldsymbol{\omega} = \int_{\partial S} \boldsymbol{\omega} = 2$.