1.3 Electromagnetism

1.3.1 Maxwell's equations

Maxwell's equations are naturally expressed in terms of differential forms

$$\begin{aligned} \underline{\nabla} \wedge \underline{\underline{B}} &= 0 \quad , \quad \underline{\nabla} \wedge \underline{E} + \dot{\underline{\underline{B}}} = 0 \\ \underline{\nabla} \wedge \underline{\underline{D}} &= \underline{\underline{\underline{\rho}}} \quad , \quad \underline{\nabla} \wedge \underline{H} - \dot{\underline{\underline{D}}} = \underline{\underline{j}} \end{aligned} \tag{1.3.1}$$

The electric displacement field (electric flux density) $\underline{\underline{D}}$ is related to the electric field $\underline{E}$ by

$$\underline{\underline{D}} = \underline{\underline{\vec{\varepsilon}_0}} \cdot \underline{E} + \underline{\underline{P}} \tag{1.3.2}$$

where $\underline{\underline{P}}$ is the polarization (electric dipole) density, and the magnetic field $\underline{H}$ is related to the magnetic flux density $\underline{\underline{B}}$ by

$$\underline{H} = \underline{\vec{\vec{\mu}}_0}^{-1} \cdot \underline{\underline{B}} - \underline{M} \tag{1.3.3}$$

where $\underline{M}$ is the magnetization (magnetic dipole) density. [1]

The electromagnetic energy density is

$$\underline{\underline{\underline{u}}} = \frac{1}{2} \underline{E} \wedge \underline{\underline{D}} + \frac{1}{2} \underline{\underline{B}} \wedge \underline{H} \tag{1.3.4}$$

and the Poynting vector (electromagnetic energy flux density) is

$$\underline{\underline{S}} = \underline{E} \wedge \underline{H} \tag{1.3.5}$$

The first line of Eq. (1.3.1) is automatic given

$$\underline{E} = -\underline{\nabla} \wedge \phi - \dot{\underline{A}} \tag{1.3.6}$$

$$\underline{\underline{B}} = \underline{\nabla} \wedge \underline{A} \tag{1.3.7}$$

which has gauge invariance

$$\phi \to \phi + \dot{\lambda} \tag{1.3.8}$$

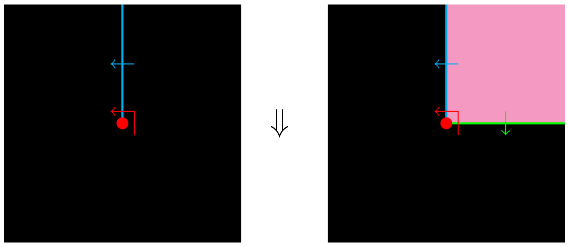
$$\underline{A} \to \underline{A} - \underline{\nabla} \wedge \lambda \tag{1.3.9}$$

while the second line of Eq. (1.3.1) gives

$$\dot{\underline{\underline{\underline{\rho}}}} + \underline{\nabla} \wedge \underline{\underline{j}} = 0 \tag{1.3.10}$$

[1] The tensors $\underline{\underline{\vec{\varepsilon}_0}}$ and $\underline{\vec{\vec{\mu}}_0}^{-1}$ will be discussed later.

$$\underline{\underline{B}} = \underline{\nabla} \wedge \underline{A} \qquad \& \qquad \underline{E} = -\underline{\nabla}\phi - \frac{\partial}{\partial t}\underline{A}$$



$$\underline{\nabla} \wedge \underline{\underline{B}} = 0 \qquad \& \qquad \underline{\nabla} \wedge \underline{E} + \frac{\partial}{\partial t}\underline{\underline{B}} = 0$$

Figure 1.3.1: Electric field and magnetic flux induced by electric and magnetic potentials. One internal space dimension has been suppressed.

$$\underline{\nabla} \wedge \underline{\underline{D}} = \underline{\underline{\underline{\rho}}} \qquad \& \qquad \underline{\nabla} \wedge \underline{H} - \frac{\partial}{\partial t}\underline{\underline{D}} = \underline{\underline{j}}$$

$$\Longrightarrow \qquad \frac{\partial}{\partial t}\underline{\underline{\underline{\rho}}} + \underline{\nabla} \wedge \underline{\underline{j}} = 0$$

Figure 1.3.2: Electric flux and magnetic field induced by charge and current. One external space dimension has been suppressed.

1.3.2 Spacetime decomposition

The 3+1 dimensional description of Maxwell's equations above requires a choice of time coordinate t in 4-dimensional spacetime. The one-form field

$$\underline{e}^t = \underline{\nabla} \wedge t \tag{1.3.11}$$

then represents the corresponding choice of 3-dimensional spatial hypersurfaces in 4-dimensional spacetime. One can also introduce a vector field $\vec{e}_t$ satisfying

$$\vec{e}_t \cdot \underline{e}^t = 1 \tag{1.3.12}$$

corresponding to the 1-dimensional time lines that define the spatial rest-frame. A spacetime displacement then decomposes as

$$\vec{dx} = dt\, \vec{e}_t + \vec{dx}_3 \tag{1.3.13}$$

where the spatial displacement satifies

$$\underline{e}^t \cdot \vec{dx}_3 = 0 \tag{1.3.14}$$

i.e. spatial displacements lie within the spatial hypersurfaces.

More generally, an n-form $\boldsymbol{\omega}$ can be decomposed into an $(n-1)$-form $\boldsymbol{\omega}^{(1)}$ and an n-form $\boldsymbol{\omega}^{(3)}$ as

$$\boldsymbol{\omega} = \underline{e}^t \wedge \boldsymbol{\omega}^{(1)} + \boldsymbol{\omega}^{(3)} \tag{1.3.15}$$

with

$$\vec{e}_t \cdot \boldsymbol{\omega}^{(1)} = 0 \tag{1.3.16}$$

$$\vec{e}_t \cdot \boldsymbol{\omega}^{(3)} = 0 \tag{1.3.17}$$

and similarly for n-vectors. Inverting gives

$$\boldsymbol{\omega}^{(1)} = \vec{e}_t \cdot \boldsymbol{\omega} \tag{1.3.18}$$

$$\boldsymbol{\omega}^{(3)} = \vec{e}_t \cdot \left(\underline{e}^t \wedge \boldsymbol{\omega}\right) \tag{1.3.19}$$

and so we can also express this decomposition as

$$\boldsymbol{\omega} = \underline{e}^t \wedge (\vec{e}_t \cdot \boldsymbol{\omega}) + \vec{e}_t \cdot \left(\underline{e}^t \wedge \boldsymbol{\omega}\right) \tag{1.3.20}$$

For example, the spacetime current density decomposes as

$$\underline{\underline{\underline{J}}} = \underline{e}^t \wedge \underline{\underline{j}} - \underline{\underline{\underline{\rho}}} \tag{1.3.21}$$

where the spatial current density and charge density

$$\underline{\underline{j}} = \vec{e}_t \cdot \underline{\underline{\underline{J}}} \tag{1.3.22}$$

$$\underline{\underline{\underline{\rho}}} = -\vec{e}_t \cdot \left(\underline{e}^t \wedge \underline{\underline{\underline{J}}}\right) \tag{1.3.23}$$

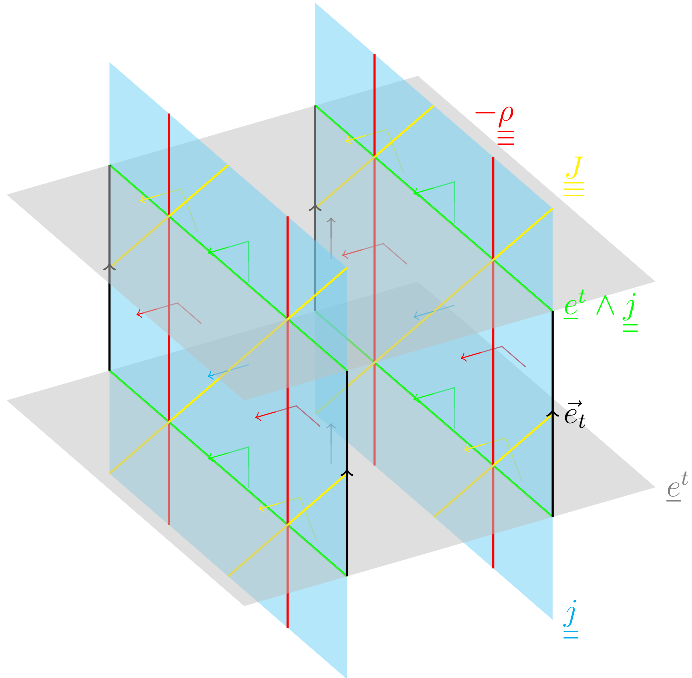


Figure 1.3.3: An illustration of the 3+1 decomposition of $\underline{\underline{\underline{J}}}$ given in Eq. (1.3.21). One spatial dimension, and densities and orientations in that dimension, has been suppressed. See J.pdf for a rotatable version of this figure.

satisfy

$$\vec{e}_t \cdot \underline{\underline{j}} = 0 \tag{1.3.24}$$

$$\vec{e}_t \cdot \underline{\underline{\underline{\rho}}} = 0 \tag{1.3.25}$$

see Figure 1.3.3.

Defining the time and spatial exterior derivatives

$$\dot{\boldsymbol{\omega}} \equiv \mathcal{L}_{e_t}\boldsymbol{\omega} = \vec{e}_t \cdot (\underline{\nabla} \wedge \boldsymbol{\omega}) + \underline{\nabla} \wedge (\vec{e}_t \cdot \boldsymbol{\omega}) \tag{1.3.26}$$

$$\underline{\nabla}^{(3)} \wedge \boldsymbol{\omega} \equiv \vec{e}_t \cdot \left(\underline{e}^t \wedge \underline{\nabla} \wedge \boldsymbol{\omega}\right) - \underline{e}^t \wedge \underline{\nabla} \wedge (\vec{e}_t \cdot \boldsymbol{\omega}) \tag{1.3.27}$$

and using Eq. (1.3.20), the spacetime exterior derivative decomposes as

$$\underline{\nabla} \wedge \boldsymbol{\omega} = \underline{e}^t \wedge \dot{\boldsymbol{\omega}} + \underline{\nabla}^{(3)} \wedge \boldsymbol{\omega} \tag{1.3.28}$$

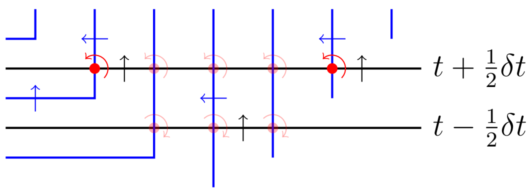


Figure 1.3.4: The topological interpretation of $\underline{e}^t \wedge \dot{\boldsymbol{\omega}}$ is given by the intersection of $\boldsymbol{\omega}$'s surfaces with the surface $t + \frac{1}{2}\delta t$ minus the intersection with the surface $t - \frac{1}{2}\delta t$, divided by δt.

Now

$$\underline{\nabla} \wedge \underline{e}^t = \underline{\nabla} \wedge \underline{\nabla} \wedge t = 0 \tag{1.3.29}$$

therefore, using Eqs. (1.3.15), (1.2.5) and (1.3.28),

$$\begin{aligned} \underline{\nabla} \wedge \boldsymbol{\omega} &= \underline{\nabla} \wedge \left(\underline{e}^t \wedge \boldsymbol{\omega}^{(1)} + \boldsymbol{\omega}^{(3)}\right) && (1.3.30) \\ &= \left(\underline{\nabla} \wedge \underline{e}^t\right) \wedge \boldsymbol{\omega}^{(1)} - \underline{e}^t \wedge \underline{\nabla} \wedge \boldsymbol{\omega}^{(1)} + \underline{\nabla} \wedge \boldsymbol{\omega}^{(3)} && (1.3.31) \\ &= -\underline{e}^t \wedge \underline{\nabla}^{(3)} \wedge \boldsymbol{\omega}^{(1)} + \underline{e}^t \wedge \dot{\boldsymbol{\omega}}^{(3)} + \underline{\nabla}^{(3)} \wedge \boldsymbol{\omega}^{(3)} && (1.3.32) \end{aligned}$$

1.3.3 Relativistic electrodynamics

Maxwell's equations

To relativize Eqs. (1.3.6) and (1.3.7), we introduce the electromagnetic potential

$$\underline{A} = \phi \underline{e}^t - \underline{A}^{(3)} \tag{1.3.33}$$

and electromagnetic field strength

$$\underline{\underline{F}} = \underline{e}^t \wedge \underline{E} - \underline{\underline{B}} \tag{1.3.34}$$

then, using Eq. (1.3.32),

$$\underline{\underline{F}} = \underline{\nabla} \wedge \underline{A} \tag{1.3.35}$$

decomposes to

$$\underline{e}^t \wedge \underline{E} - \underline{\underline{B}} = \underline{e}^t \wedge \left(-\underline{\nabla}^{(3)}\phi - \dot{\underline{A}}^{(3)}\right) - \underline{\nabla}^{(3)} \wedge \underline{A}^{(3)} \tag{1.3.36}$$

relativizing Eqs. (1.3.6) and (1.3.7). Eq. (1.3.35) has gauge invariance

$$\underline{A} \to \underline{A} + \underline{\nabla} \wedge \lambda \tag{1.3.37}$$

which decomposes to

$$\phi \underline{e}^t - \underline{A}^{(3)} \to \left(\phi + \dot{\lambda}\right) \underline{e}^t - \left(\underline{A}^{(3)} - \underline{\nabla}^{(3)}\lambda\right) \tag{1.3.38}$$

relativizing Eqs. (1.3.8) and (1.3.9), and implies

$$\underline{\nabla} \wedge \underline{\underline{F}} = 0 \tag{1.3.39}$$

which decomposes to

$$-\underline{e}^t \wedge \left(\underline{\nabla}^{(3)} \wedge \underline{E} + \dot{\underline{\underline{B}}}\right) - \underline{\nabla}^{(3)} \wedge \underline{\underline{B}} = 0 \tag{1.3.40}$$

relativizing the first line of Maxwell's equations, Eq. (1.3.1).

To get the second line of Maxwell's equations, we introduce the electromagnetic flux density

$$\underline{\underline{G}} = -\underline{e}^t \wedge \underline{H} - \underline{\underline{D}} \tag{1.3.41}$$

and spacetime current density, Eq. (1.3.21),

$$\underline{\underline{\underline{J}}} = \underline{e}^t \wedge \underline{\underline{j}} - \underline{\underline{\underline{\rho}}} \tag{1.3.42}$$

Then, using Eq. (1.3.32),

$$\underline{\nabla} \wedge \underline{\underline{G}} = \underline{\underline{\underline{J}}} \tag{1.3.43}$$

decomposes to

$$\underline{e}^t \wedge \left(\underline{\nabla}^{(3)} \wedge \underline{H} - \dot{\underline{\underline{D}}}\right) - \underline{\nabla}^{(3)} \wedge \underline{\underline{D}} = \underline{e}^t \wedge \underline{\underline{j}} - \underline{\underline{\underline{\rho}}} \tag{1.3.44}$$

relativizing the second line of Maxwell's equations, Eq. (1.3.1). Eq. (1.3.43) implies

$$\underline{\nabla} \wedge \underline{\underline{\underline{J}}} = 0 \tag{1.3.45}$$

which decomposes to

$$-\underline{e}^t \wedge \left(\dot{\underline{\underline{\underline{\rho}}}} + \underline{\nabla}^{(3)} \wedge \underline{\underline{j}}\right) = 0 \tag{1.3.46}$$

relativizing Eq. (1.3.10).

In summary, the relativistic form of Maxwell's equations is

$$\underline{\nabla} \wedge \underline{\underline{F}} = 0 \tag{1.3.47}$$

$$\underline{\nabla} \wedge \underline{\underline{G}} = \underline{\underline{\underline{J}}} \tag{1.3.48}$$

with Eq. (1.3.47) implied by

$$\underline{\underline{F}} = \underline{\nabla} \wedge \underline{A} \tag{1.3.49}$$

which has gauge invariance

$$\underline{A} \to \underline{A} + \underline{\nabla} \wedge \lambda \tag{1.3.50}$$

and Eq. (1.3.48) implying

$$\underline{\nabla} \wedge \underline{\underline{\underline{J}}} = 0 \tag{1.3.51}$$

These equations, and their implication of the equations in Section 1.3.1, are illustrated in Figures 1.3.5 and 1.3.6.

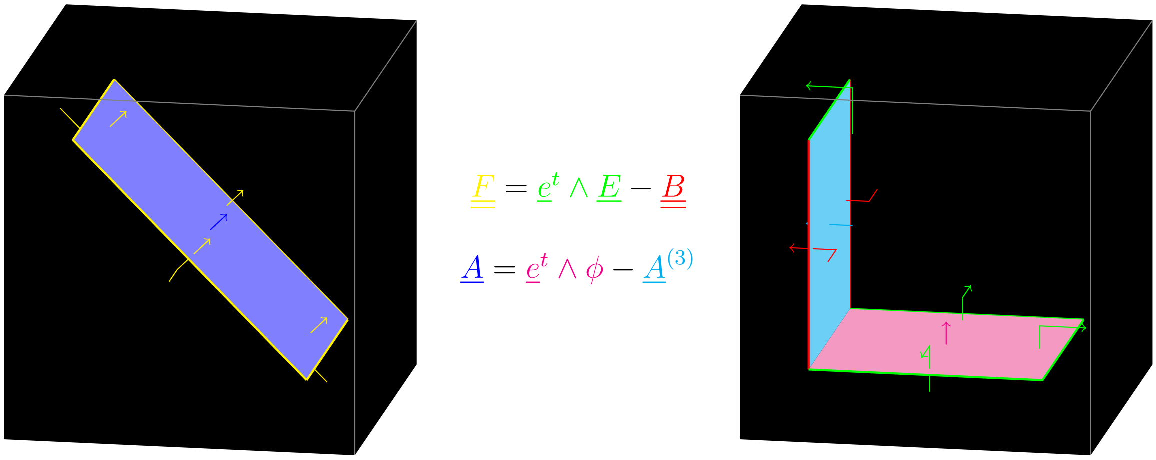


Figure 1.3.5: Space and time decomposition of $\underline{\underline{F}} = \underline{\nabla} \wedge \underline{A}$ and $\underline{\nabla} \wedge \underline{\underline{F}} = 0$ to give $\underline{\underline{B}} = \underline{\nabla}^{(3)} \wedge \underline{A}^{(3)}$ and $\underline{e}^t \wedge \underline{E} = \underline{\nabla}^{(3)} \wedge (\underline{e}^t \wedge \phi) - \underline{e}^t \wedge \frac{\partial}{\partial t}\underline{A}^{(3)}$ and $\underline{\nabla}^{(3)} \wedge \underline{\underline{B}} = 0$ and $-\underline{\nabla}^{(3)} \wedge (\underline{e}^t \wedge \underline{E}) + \underline{e}^t \wedge \frac{\partial}{\partial t}\underline{\underline{B}} = 0$. One internal space dimension has been suppressed. Compare Figure 1.3.1.

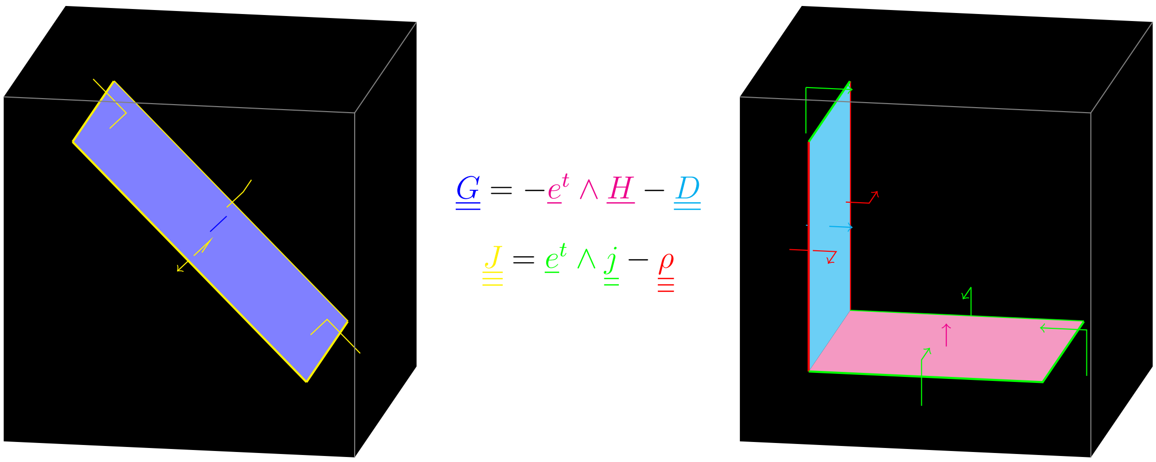


Figure 1.3.6: Space and time decomposition of $\underline{\nabla} \wedge \underline{\underline{G}} = \underline{\underline{\underline{J}}}$ and $\underline{\nabla} \wedge \underline{\underline{\underline{J}}} = 0$ to give $\underline{\nabla}^{(3)} \wedge \underline{\underline{D}} = \underline{\underline{\underline{\rho}}}$ and $-\underline{\nabla}^{(3)} \wedge (\underline{e}^t \wedge \underline{H}) - \underline{e}^t \wedge \frac{\partial}{\partial t}\underline{\underline{D}} = \underline{e}^t \wedge \underline{\underline{j}}$ and $-\underline{\nabla}^{(3)} \wedge \left(\underline{e}^t \wedge \underline{\underline{j}}\right) + \underline{e}^t \wedge \frac{\partial}{\partial t}\underline{\underline{\underline{\rho}}} = 0$. One external space dimension has been suppressed. Compare Figure 1.3.2.

Lorentz force

Using the coordinate time t, we can define the spacetime velocity [2]

$$\vec{v} = \frac{d\vec{x}}{dt} \tag{1.3.52}$$

which, following Eq. (1.3.13), decomposes as

$$\vec{v} = \vec{e}_t + \vec{v}_3 \tag{1.3.53}$$

and the spacetime force

$$\underline{f} = \frac{d\underline{p}}{dt} \tag{1.3.54}$$

which decomposes as

$$\underline{f} = P\underline{e}^t - \underline{F} \tag{1.3.55}$$

Eqs. (1.3.34) and (1.3.53) give

$$\underline{\underline{F}} \cdot \vec{v} = \left(\underline{e}^t \wedge \underline{E} - \underline{\underline{B}}\right) \cdot (\vec{e}_t + \vec{v}_3) \tag{1.3.56}$$

$$= (\underline{E} \cdot \vec{v}_3)\, \underline{e}^t - \left(\underline{E} + \underline{\underline{B}} \cdot \vec{v}_3\right) \tag{1.3.57}$$

Therefore

$$\underline{f} = q\underline{\underline{F}} \cdot \vec{v} \tag{1.3.58}$$

relativizes both the electromagnetic power equation

$$P = q\underline{E} \cdot \vec{v}_3 \tag{1.3.59}$$

and the Lorentz force law

$$\underline{F} = q\left(\underline{E} + \underline{\underline{B}} \cdot \vec{v}_3\right) \tag{1.3.60}$$

[2]The proper velocity, which is defined with respect to the proper time and has unit magnitude, is independent of the choice of time coordinate but requires a definition of length and so can not be defined topologically.