1.4 Densities and volumes

1.4.1 Volume form and volume element

A **volume form** $\boldsymbol{\epsilon}$ provides a measure or oriented unit density on a space and a (unit) **volume element** $\boldsymbol{\epsilon}^{-1}$ provides an oriented unit volume. One induces the other via

$$\boldsymbol{\epsilon}^{-1} \cdot \boldsymbol{\epsilon} = 1 \tag{1.4.1}$$

For example, in three dimensions,

$$dV = \underline{\underline{\underline{\epsilon}}} \cdot d\vec{\vec{\vec{V}}} \tag{1.4.2}$$

is the volume of the infinitesimal volume element $d\vec{\vec{\vec{V}}}$ and

$$\rho = \vec{\vec{\vec{\epsilon}}} \cdot \underline{\underline{\underline{\rho}}} \tag{1.4.3}$$

is the charge per unit volume of the charge density $\underline{\underline{\underline{\rho}}}$. More generally, the volume form and element induce a **volume duality**[1] between n-vectors $\boldsymbol{v}$ and $(N-n)$-forms $\star\boldsymbol{v}$, and n-forms $\boldsymbol{\omega}$ and $(N-n)$-vectors $\star\boldsymbol{\omega}$,

$$\begin{array}{rclcrcl} \star\boldsymbol{v} & \equiv & \boldsymbol{v} \cdot \boldsymbol{\epsilon} & , & \boldsymbol{v} & = & \boldsymbol{\epsilon}^{-1} \cdot \star\boldsymbol{v} \\ \star\boldsymbol{\omega} & \equiv & \boldsymbol{\omega} \cdot \boldsymbol{\epsilon}^{-1} & , & \boldsymbol{\omega} & = & \boldsymbol{\epsilon} \cdot \star\boldsymbol{\omega} \end{array} \tag{1.4.4}$$

Using Eq. (1.1.21),

$$\begin{array}{rclclcl} \star^{-1}\boldsymbol{v} & = & \boldsymbol{\epsilon} \cdot \boldsymbol{v} & = & (-1)^{n(N-n)}\boldsymbol{v} \cdot \boldsymbol{\epsilon} & = & (-1)^{n(N-n)} \star \boldsymbol{v} \\ \star^{-1}\boldsymbol{\omega} & = & \boldsymbol{\epsilon}^{-1} \cdot \boldsymbol{\omega} & = & (-1)^{n(N-n)}\boldsymbol{\omega} \cdot \boldsymbol{\epsilon}^{-1} & = & (-1)^{n(N-n)} \star \boldsymbol{\omega} \end{array} \tag{1.4.5}$$

and so in three dimensions $\star^{-1} = \star$. For example, the traditional vector representation $\vec{B}$ of the magnetic flux density $\underline{\underline{B}}$ is

$$\vec{B} = \star\underline{\underline{B}} \tag{1.4.6}$$

1.4.2 Divergence

We can use volume duality to define the divergence of an n-vector

$$(-1)^{n-1}\boldsymbol{\nabla} \cdot \boldsymbol{v} \equiv \star^{-1}\boldsymbol{\nabla} \wedge \star\boldsymbol{v} \tag{1.4.7}$$

See Figure 1.4.1 and Eq. (Q5.3.2).

For example, in three dimensions, the divergence of the traditional vector representation $\vec{B}$ of the magnetic flux density $\underline{\underline{B}}$ is

$$\underline{\nabla} \cdot \vec{B} = \star\left(\underline{\nabla} \wedge \underline{\underline{B}}\right) \tag{1.4.8}$$

[1] This duality is similar to the Hodge $*$ duality between n-forms and $(N-n)$-forms defined in Section 2.1.3, so we use a similar notation.

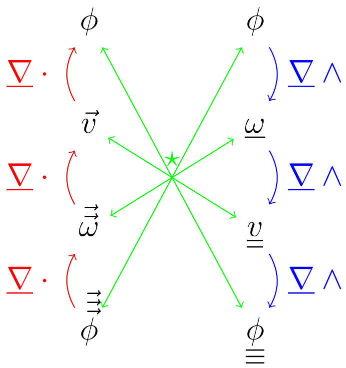

Figure 1.4.1: Volume duality in three dimensions.

1.4.3 Integration of multivectors

The volume form $\boldsymbol{\epsilon}$ provides a measure for the integration of a scalar ϕ over an N-dimensional volume V

$$\int_V \phi\,\boldsymbol{\epsilon} = \int_V \phi\,\boldsymbol{\epsilon}\cdot\boldsymbol{dV} = \int_V \phi\,dV = \text{scalar} \tag{1.4.9}$$

More generally, we can use volume duality to integrate an $(N-n)$-vector $\boldsymbol{v}$ over an n-dimensional surface S

$$\begin{aligned} & \int_S \star\boldsymbol{v} &=& \int_S (\star\boldsymbol{v})\cdot\boldsymbol{dS} &=& \int_S \boldsymbol{v}\cdot\star^{-1}\boldsymbol{dS} \\ =& \int_S \boldsymbol{v}\cdot\boldsymbol{\epsilon} &=& \int_S (\boldsymbol{v}\cdot\boldsymbol{\epsilon})\cdot\boldsymbol{dS} &=& \int_S \boldsymbol{v}\cdot\boldsymbol{\epsilon}\cdot\boldsymbol{dS} \\ =& \int_S (\boldsymbol{v}\cdot\boldsymbol{n})\,\boldsymbol{\sigma} &=& \int_S (\boldsymbol{v}\cdot\boldsymbol{n})\,\boldsymbol{\sigma}\cdot\boldsymbol{dS} &=& \int_S \boldsymbol{v}\cdot\boldsymbol{n}\,dS \end{aligned} \tag{1.4.10}$$

where $\boldsymbol{n}$ is the normal $(N-n)$-form to the surface S

$$\boldsymbol{n}\cdot\boldsymbol{dS} = 0 \tag{1.4.11}$$

$\boldsymbol{\sigma}$ is the volume form on S

$$\boldsymbol{\sigma}\cdot\boldsymbol{dS} = dS \tag{1.4.12}$$

and we have used

$$\boldsymbol{\epsilon} = \boldsymbol{n}\wedge\boldsymbol{\sigma} \tag{1.4.13}$$

or equivalently

$$\star^{-1}\boldsymbol{dS} = \boldsymbol{n}\,dS \tag{1.4.14}$$

Combining Eqs. (1.2.11) and (1.4.7) gives the **divergence theorem**

$$(-1)^{n-1}\int_S \star(\boldsymbol{\nabla}\cdot\boldsymbol{v}) = \int_{\partial S}\star\boldsymbol{v} \tag{1.4.15}$$

for an n-vector $\boldsymbol{v}$ and an $(N-n+1)$-surface S.