1.5 Bases and coordinates

1.5.1 Bases and components

It is often convenient to choose a complete set of independent **basis** vectors $\vec{e}_\alpha$, where $\alpha = 1, \ldots, N$ labels the basis vectors and N is the dimension of the space, and express a general vector $\vec{v}$ as a linear combination of the basis vectors

$$\vec{v} = \sum_{\alpha=1}^{N} v^\alpha \vec{e}_\alpha \tag{1.5.1}$$

The scalars v^α are the **components** of the vector $\vec{v}$ and depend on the choice of basis $\vec{e}_\alpha$. We will use the summation convention for repeated component indices, so that the summation sign above is not explicitly written

$$\vec{v} = v^\alpha \vec{e}_\alpha \tag{1.5.2}$$

A vector basis $\vec{e}_\alpha$ naturally induces a covector basis $\underline{e}^\alpha$, or vice versa, via

$$\underline{e}^\alpha \cdot \vec{e}_\beta = \delta^\alpha_\beta \tag{1.5.3}$$

A covector is expressed in components as

$$\underline{\omega} = \omega_\alpha \underline{e}^\alpha \tag{1.5.4}$$

and a covector contracted with a vector as

$$\underline{\omega} \cdot \vec{v} = \omega_\alpha v^\alpha \tag{1.5.5}$$

In three dimensions, a 2-form is expressed in terms of a **differential form basis** as

$$\begin{aligned} \underline{\underline{\omega}} &= \omega_{12}\, \underline{e}^1 \wedge \underline{e}^2 + \omega_{23}\, \underline{e}^2 \wedge \underline{e}^3 + \omega_{31}\, \underline{e}^3 \wedge \underline{e}^1 && (1.5.6) \\ &= \frac{1}{2!} \omega_{\alpha\beta}\, \underline{e}^\alpha \wedge \underline{e}^\beta && (1.5.7) \end{aligned}$$

where the factorial is needed to compensate for the index permutations. More generally, an n-form $\boldsymbol{\omega}$ is expressed as

$$\boldsymbol{\omega} = \frac{1}{n!} \omega_{\alpha_1 \cdots \alpha_n}\, \underline{e}^{\alpha_1} \wedge \ldots \wedge \underline{e}^{\alpha_n} \tag{1.5.8}$$

and similarly for multivectors. The differential form and multivector bases are orthonormal to each other, generalizing Eq. (1.5.3),

$$(\underline{e}^{\alpha_1} \wedge \ldots \wedge \underline{e}^{\alpha_n}) \cdot (\vec{e}_{\beta_1} \wedge \ldots \wedge \vec{e}_{\beta_n}) = \delta^{\alpha_1 \cdots \alpha_n}_{\beta_1 \cdots \beta_n} \tag{1.5.9}$$

and more generally

$$(\underline{e}^{\alpha_1} \wedge \ldots \wedge \underline{e}^{\alpha_m}) \cdot (\vec{e}_{\beta_1} \wedge \ldots \wedge \vec{e}_{\beta_n}) = \begin{cases} \dfrac{1}{(n-m)!} \delta^{\alpha_1 \cdots \alpha_m \gamma_{m+1} \cdots \gamma_n}_{\beta_1 \cdots \beta_n}\, \vec{e}_{\gamma_{m+1}} \wedge \ldots \wedge \vec{e}_{\gamma_n} & \text{for } m \leq n \\ \dfrac{1}{(m-n)!}\, \underline{e}^{\gamma_{n+1}} \wedge \ldots \wedge \underline{e}^{\gamma_m}\, \delta^{\alpha_1 \cdots \alpha_m}_{\gamma_{n+1} \cdots \gamma_m \beta_1 \cdots \beta_n} & \text{for } m \geq n \end{cases} \tag{1.5.10}$$

where the **generalized Kronecker delta** $\delta^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n}$ has the property

$$\frac{1}{n!}\delta^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n}\omega_{\alpha_1\cdots\alpha_n} = \omega_{\beta_1\cdots\beta_n} \tag{1.5.11}$$

and is antisymmetric with respect to both sets of indices

$$\delta^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n} = \begin{cases} \delta^{\alpha_1}_{\beta_1} & \text{for } n=1 \\ \delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2} - \delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_1} & \text{for } n=2 \\ \delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}\delta^{\alpha_3}_{\beta_3} + \delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_3}\delta^{\alpha_3}_{\beta_1} + \delta^{\alpha_1}_{\beta_3}\delta^{\alpha_2}_{\beta_1}\delta^{\alpha_3}_{\beta_2} & \\ \quad - \delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_3}\delta^{\alpha_3}_{\beta_2} - \delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_1}\delta^{\alpha_3}_{\beta_3} - \delta^{\alpha_1}_{\beta_3}\delta^{\alpha_2}_{\beta_2}\delta^{\alpha_3}_{\beta_1} & \text{for } n=3 \end{cases} \tag{1.5.12}$$

The volume form has a single component

$$\boldsymbol{\epsilon} = \epsilon_{1\cdots N}\, \underline{e}^1 \wedge \ldots \wedge \underline{e}^N \tag{1.5.13}$$

The magnitude of

$$\epsilon_{1\cdots N} = \boldsymbol{\epsilon}\cdot(\vec{e}_1 \wedge \ldots \wedge \vec{e}_N) \tag{1.5.14}$$

is the physical volume of the basis volume element $\vec{e}_1 \wedge \ldots \wedge \vec{e}_N$, and its sign corresponds to the orientation of the basis relative to that of the space, and is conventionally fixed by taking $\epsilon_{1\cdots N} > 0$. Eqs. (1.4.1) and (1.5.13) give

$$\boldsymbol{\epsilon}^{-1} = \frac{1}{\epsilon_{1\cdots N}}\, \vec{e}_1 \wedge \ldots \wedge \vec{e}_N \tag{1.5.15}$$

The components of the volume element and form are related to the generalised Kronecker delta via the Levi-Civita identities

$$\frac{1}{(N-n)!}\epsilon^{-1\alpha_1\cdots\alpha_n\gamma_{n+1}\cdots\gamma_N}\epsilon_{\beta_1\cdots\beta_n\gamma_{n+1}\cdots\gamma_N} = \delta^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n} \tag{1.5.16}$$

1.5.2 Coordinate bases

A coordinate system x^α induces a covector **coordinate basis** via

$$\underline{e}^\alpha = \underline{\nabla} \wedge x^\alpha \tag{1.5.17}$$

and the corresponding vector coordinate basis induced by Eq. (1.5.3) expands an infinitesimal displacement as

$$\vec{dx} = dx^\alpha \vec{e}_\alpha \tag{1.5.18}$$

where dx^α is the infinitesimal change in the coordinate x^α. Inverting Eq. (1.5.18) gives

$$\vec{e}_\alpha = \frac{\partial\vec{x}}{\partial x^\alpha} \tag{1.5.19}$$

Note that a coordinate basis covector $\underline{e}^\alpha$ is defined purely in terms of its coordinate x^α, with its plane tangent to the constant x^α surfaces and its magnitude given by the density of the surfaces, while a coordinate basis vector $\vec{e}_\alpha$ requires the full coordinate system, with its line tangent to the intersection of the constant x^β, $\beta \neq \alpha$, surfaces and its magnitude given by the separation of the x^α surfaces. See Figure 1.5.1.

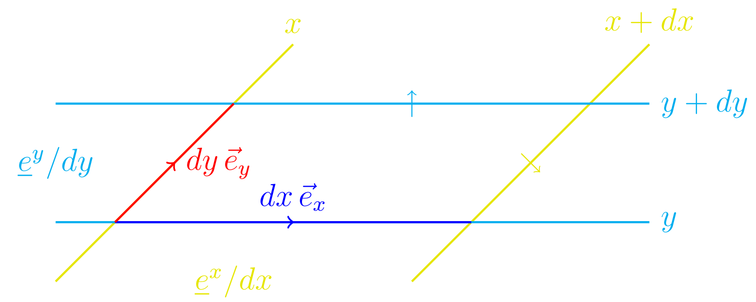

Figure 1.5.1: The coordinate basis covectors $\underset{\sim}{e}^x$ and $\underset{\sim}{e}^y$ are given by the x and y contours respectively. The coordinate basis vectors $\vec{e}_x$ and $\vec{e}_y$ lie along the y and x contours respectively, and span the x and y contours respectively.

Exterior derivative in a coordinate basis

In a coordinate basis

$$\underset{\sim}{e}^\alpha = \underset{\sim}{\nabla} \wedge x^\alpha \tag{1.5.20}$$

and so

$$\underset{\sim}{\nabla} \wedge \underset{\sim}{e}^\alpha = \underset{\sim}{\nabla} \wedge \underset{\sim}{\nabla} \wedge x^\alpha = 0 \tag{1.5.21}$$

therefore the exterior derivative of an n-form $\boldsymbol{\omega}$ in a coordinate basis is

$$\underset{\sim}{\nabla} \wedge \boldsymbol{\omega} = \frac{1}{n!}\frac{\partial \omega_{\beta_1\cdots\beta_n}}{\partial x^\alpha}\underset{\sim}{e}^\alpha \wedge \underset{\sim}{e}^{\beta_1} \wedge \ldots \wedge \underset{\sim}{e}^{\beta_n} \tag{1.5.22}$$

For example, in three dimensions

$$\underset{\sim}{\nabla} \wedge \phi = \frac{\partial \phi}{\partial x^1}\underset{\sim}{e}^1 + \frac{\partial \phi}{\partial x^2}\underset{\sim}{e}^2 + \frac{\partial \phi}{\partial x^3}\underset{\sim}{e}^3 \tag{1.5.23}$$

$$\underset{\sim}{\nabla} \wedge \underset{\sim}{\omega} = \left(\frac{\partial \omega_2}{\partial x^1} - \frac{\partial \omega_1}{\partial x^2}\right)\underset{\sim}{e}^1 \wedge \underset{\sim}{e}^2 + \left(\frac{\partial \omega_3}{\partial x^2} - \frac{\partial \omega_2}{\partial x^3}\right)\underset{\sim}{e}^2 \wedge \underset{\sim}{e}^3 + \left(\frac{\partial \omega_1}{\partial x^3} - \frac{\partial \omega_3}{\partial x^1}\right)\underset{\sim}{e}^3 \wedge \underset{\sim}{e}^1 \tag{1.5.24}$$

$$\underset{\sim}{\nabla} \wedge \underset{\approx}{\omega} = \left(\frac{\partial \omega_{23}}{\partial x^1} + \frac{\partial \omega_{31}}{\partial x^2} + \frac{\partial \omega_{12}}{\partial x^3}\right)\underset{\sim}{e}^1 \wedge \underset{\sim}{e}^2 \wedge \underset{\sim}{e}^3 \tag{1.5.25}$$

Integration in a coordinate basis

The integral of an n-form

$$\boldsymbol{\omega} = \frac{1}{n!}\omega_{\alpha_1\cdots\alpha_n}\,\underset{\sim}{e}^{\alpha_1} \wedge \ldots \wedge \underset{\sim}{e}^{\alpha_n} \tag{1.5.26}$$

over an n-surface S with infinitesimal surface element

$$\boldsymbol{dS} = \frac{1}{n!}\,dS^{\alpha_1\cdots\alpha_n}\,\vec{e}_{\alpha_1} \wedge \ldots \wedge \vec{e}_{\alpha_n} \tag{1.5.27}$$

is

$$\int_S \boldsymbol{\omega} = \int_S \boldsymbol{\omega} \cdot \boldsymbol{dS} = \int_S \frac{1}{n!}\omega_{\alpha_1\cdots\alpha_n}\,dS^{\alpha_1\cdots\alpha_n} \tag{1.5.28}$$

If we choose coordinates x^α such that $x^{n+1}, \ldots, x^N$ are constant on S, then

$$dS^{1\cdots n} = dx^1 \ldots dx^n \tag{1.5.29}$$

and Eq. (1.5.28) simplifies to

$$\int_S \boldsymbol{\omega} = \int_S \omega_{1\cdots n}\, dx^1 \ldots dx^n \tag{1.5.30}$$

Similarly, the integral of a scalar ϕ over a volume V becomes

$$\int_V \phi\, \boldsymbol{\epsilon} = \int_V \phi\, \boldsymbol{\epsilon} \cdot \boldsymbol{dV} = \int_V \phi\, \epsilon_{1\ldots N}\, dx^1 \ldots dx^N \tag{1.5.31}$$