2.2 Tensor calculus

2.2.1 Covariant derivative

The **covariant derivative** $\nabla_{\mathbf{a}}$ is a derivative operator and hence is linear and obeys the Leibnitz rule. Its action on a scalar is given by Eq. (1.1.8)

$$dx^{\mathbf{a}}\nabla_{\mathbf{a}}\phi = d\phi \tag{2.2.1}$$

and it has the key property

$$\nabla_{\mathbf{a}} g_{\mathbf{bc}} = 0 \tag{2.2.2}$$

since the metric is used to measure changes. One more condition is needed to uniquely define the covariant derivative, the zero torsion condition

$$\left(\nabla_{\mathbf{a}}\nabla_{\mathbf{b}} - \nabla_{\mathbf{b}}\nabla_{\mathbf{a}}\right)\phi = 0 \tag{2.2.3}$$

which has the geometrical interpretation that parallel transported vectors, i.e. vectors transported such that their covariant derivative is zero, form closed parallelograms.

Covariant derivatives do not commute when acting on tensors

$$\left(\nabla_{\mathbf{a}}\nabla_{\mathbf{b}} - \nabla_{\mathbf{b}}\nabla_{\mathbf{a}}\right) w_{\mathbf{c}} = R_{\mathbf{abc}}{}^{\mathbf{d}} w_{\mathbf{d}} \tag{2.2.4}$$

since parallel transport of tensors in a curved space is path dependent. $R_{\mathbf{abcd}}$ is the **curvature tensor**, which is zero if and only if the space is flat.

In a coordinate basis, the covariant derivative acting on a scalar has components

$$\nabla_{\mathbf{a}}\phi = \frac{\partial\phi}{\partial x^{\alpha}} e^{\alpha}_{\mathbf{a}} \tag{2.2.5}$$

Acting on a vector

$$\nabla_{\mathbf{a}} v^{\mathbf{c}} = \nabla_{\mathbf{a}}\left(v^{\beta} e^{\mathbf{c}}_{\beta}\right) = \left(\nabla_{\mathbf{a}} v^{\beta}\right) e^{\mathbf{c}}_{\beta} + v^{\beta}\nabla_{\mathbf{a}} e^{\mathbf{c}}_{\beta} \tag{2.2.6}$$

$\nabla_{\mathbf{a}} v^{\beta}$ is given by Eq. (2.2.5) and we define

$$\nabla_{\mathbf{a}} e^{\mathbf{c}}_{\beta} \equiv \Gamma^{\mathbf{c}}_{\mathbf{a}\beta} = \Gamma^{\gamma}_{\alpha\beta} e^{\alpha}_{\mathbf{a}} e^{\mathbf{c}}_{\gamma} \tag{2.2.7}$$

where the $\Gamma^{\gamma}_{\alpha\beta}$ are the **Christoffel symbols** [1]. Then

$$\nabla_{\mathbf{a}} v^{\mathbf{c}} = \left(\frac{\partial v^{\gamma}}{\partial x^{\alpha}} + \Gamma^{\gamma}_{\alpha\beta} v^{\beta}\right) e^{\alpha}_{\mathbf{a}} e^{\mathbf{c}}_{\gamma} \tag{2.2.8}$$

$$\nabla_{\mathbf{a}} \omega_{\mathbf{b}} = \left(\frac{\partial \omega_{\beta}}{\partial x^{\alpha}} - \Gamma^{\gamma}_{\alpha\beta} \omega_{\gamma}\right) e^{\alpha}_{\mathbf{a}} e^{\beta}_{\mathbf{b}} \tag{2.2.9}$$

and similarly for other tensors. Applying this to Eq. (2.2.2) we get an equation involving the first derivatives of the metric components and the Christoffel symbols which can be inverted to give

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta}\left(\frac{\partial g_{\delta\alpha}}{\partial x^{\beta}} + \frac{\partial g_{\delta\beta}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\delta}}\right) \tag{2.2.10}$$

[1] Note that $\Gamma^{\mathbf{c}}_{\mathbf{ab}} = \Gamma^{\gamma}_{\alpha\beta} e^{\alpha}_{\mathbf{a}} e^{\beta}_{\mathbf{b}} e^{\mathbf{c}}_{\gamma}$ is a tensor derived from a particular basis and so is a basis dependent tensor.

2.2.2 Acceleration and geodesics

The motion of a particle in a space is described by a curve $x(t)$. The particle's velocity

$$v^{\mathbf{a}} = \frac{dx^{\mathbf{a}}}{dt} = \frac{dx^{\alpha}}{dt} e_{\alpha}^{\mathbf{a}} \tag{2.2.11}$$

is the tangent vector to the curve. The particle's **acceleration** is

$$a^{\mathbf{a}} = \frac{dv^{\mathbf{a}}}{dt} = \left(\frac{dv^{\alpha}}{dt} + \Gamma^{\alpha}_{\beta\gamma} v^{\beta} v^{\gamma} \right) e_{\alpha}^{\mathbf{a}} \tag{2.2.12}$$

where we have used

$$\frac{de_{\alpha}^{\mathbf{a}}}{dt} = v^{\mathbf{b}} \nabla_{\mathbf{b}} e_{\alpha}^{\mathbf{a}} = v^{\beta} \Gamma^{\gamma}_{\beta\alpha} e_{\gamma}^{\mathbf{a}} \tag{2.2.13}$$

Note that the velocity is the ratio of the infinitesimal vector $dx^{\mathbf{a}}$ and the infinitesimal scalar dt, while the acceleration is the derivative of the vector $v^{\mathbf{a}}$. If the acceleration is proportional to the velocity then the particle follows a **geodesic**, the generalization of a straight line to curved spaces. Further, if the parameterization of the curve is chosen such that the acceleration is zero then the parameter is **affine** and is proportional to the length along the curve. Thus an affinely parameterized geodesic $x(s)$ obeys the **geodesic equation**

$$\frac{d^2 x^{\mathbf{a}}}{ds^2} = \left(\frac{d^2 x^{\alpha}}{ds^2} + \Gamma^{\alpha}_{\beta\gamma} \frac{dx^{\beta}}{ds} \frac{dx^{\gamma}}{ds} \right) e_{\alpha}^{\mathbf{a}} = 0 \tag{2.2.14}$$

2.2.3 Covariant partial derivatives

The **covariant partial derivatives** $\partial/\partial x^{\mathbf{a}}$ and $\partial/\partial \dot{x}^{\mathbf{a}}$ are derivatives at fixed velocity and position respectively. Acting on a (possibly tensorial) function purely of position, $\partial/\partial x^{\mathbf{a}}$ reduces to the covariant derivative $\nabla_{\mathbf{a}}$ while $\partial/\partial \dot{x}^{\mathbf{a}}$ gives zero. They have the basic properties

$$\begin{aligned} \frac{\partial x^{\mathbf{b}}}{\partial x^{\mathbf{a}}} &= \delta^{\mathbf{b}}_{\mathbf{a}} \quad , \quad \frac{\partial \dot{x}^{\mathbf{b}}}{\partial x^{\mathbf{a}}} = 0 \\ \frac{\partial x^{\mathbf{b}}}{\partial \dot{x}^{\mathbf{a}}} &= 0 \quad , \quad \frac{\partial \dot{x}^{\mathbf{b}}}{\partial \dot{x}^{\mathbf{a}}} = \delta^{\mathbf{b}}_{\mathbf{a}} \end{aligned} \tag{2.2.15}$$

We can use the chain rule to express the covariant partial derivatives in terms of the coordinate partial derivatives $\partial/\partial x^{\alpha}$ and $\partial/\partial \dot{x}^{\alpha}$

$$\frac{\partial}{\partial x^{\mathbf{a}}} = \frac{\partial x^{\alpha}}{\partial x^{\mathbf{a}}} \frac{\partial}{\partial x^{\alpha}} + \frac{\partial \dot{x}^{\alpha}}{\partial x^{\mathbf{a}}} \frac{\partial}{\partial \dot{x}^{\alpha}} \tag{2.2.16}$$

$$\frac{\partial}{\partial \dot{x}^{\mathbf{a}}} = \frac{\partial x^{\alpha}}{\partial \dot{x}^{\mathbf{a}}} \frac{\partial}{\partial x^{\alpha}} + \frac{\partial \dot{x}^{\alpha}}{\partial \dot{x}^{\mathbf{a}}} \frac{\partial}{\partial \dot{x}^{\alpha}} \tag{2.2.17}$$

with

$$\frac{\partial x^\alpha}{\partial x^{\mathbf{a}}} = \nabla_{\mathbf{a}} x^\alpha = e^\alpha_{\mathbf{a}} \tag{2.2.18}$$

$$\frac{\partial \dot{x}^\alpha}{\partial x^{\mathbf{a}}} = \frac{\partial}{\partial x^{\mathbf{a}}}\left(\dot{x}^{\mathbf{b}} e^\alpha_{\mathbf{b}}\right) = \dot{x}^{\mathbf{b}} \nabla_{\mathbf{a}} \nabla_{\mathbf{b}} x^\alpha = \dot{x}^{\mathbf{b}} \nabla_{\mathbf{b}} \nabla_{\mathbf{a}} x^\alpha = \dot{e}^\alpha_{\mathbf{a}} \tag{2.2.19}$$

$$\frac{\partial x^\alpha}{\partial \dot{x}^{\mathbf{a}}} = 0 \tag{2.2.20}$$

$$\frac{\partial \dot{x}^\alpha}{\partial \dot{x}^{\mathbf{a}}} = \frac{\partial}{\partial \dot{x}^{\mathbf{a}}}\left(\dot{x}^{\mathbf{b}} e^\alpha_{\mathbf{b}}\right) = \frac{\partial \dot{x}^{\mathbf{b}}}{\partial \dot{x}^{\mathbf{a}}} e^\alpha_{\mathbf{b}} = e^\alpha_{\mathbf{a}} \tag{2.2.21}$$

Therefore

$$\frac{\partial}{\partial x^{\mathbf{a}}} = e^\alpha_{\mathbf{a}} \frac{\partial}{\partial x^\alpha} + \dot{e}^\alpha_{\mathbf{a}} \frac{\partial}{\partial \dot{x}^\alpha} \tag{2.2.22}$$

$$\frac{\partial}{\partial \dot{x}^{\mathbf{a}}} = e^\alpha_{\mathbf{a}} \frac{\partial}{\partial \dot{x}^\alpha} \tag{2.2.23}$$

or inverting

$$\frac{\partial}{\partial x^\alpha} = e^{\mathbf{a}}_\alpha \frac{\partial}{\partial x^{\mathbf{a}}} + \dot{e}^{\mathbf{a}}_\alpha \frac{\partial}{\partial \dot{x}^{\mathbf{a}}} \tag{2.2.24}$$

$$\frac{\partial}{\partial \dot{x}^\alpha} = e^{\mathbf{a}}_\alpha \frac{\partial}{\partial \dot{x}^{\mathbf{a}}} \tag{2.2.25}$$

Note that $\partial/\partial x^{\mathbf{a}}$ and $\partial/\partial x^\alpha$ are derivatives at fixed velocity $\dot{x}^{\mathbf{a}}$ and velocity component $\dot{x}^\alpha$ respectively, hence their complicated relationship, while both $\partial/\partial \dot{x}^{\mathbf{a}}$ and $\partial/\partial \dot{x}^\alpha$ are derivatives at fixed position, hence their simple relationship.