2.3 Calculus of variations

2.3.1 Euler-Lagrange equation

The action functional

$$S[x(t)] = \int_{t_\mathrm{i}}^{t_\mathrm{f}} L(x, \dot{x}, t)\, dt \tag{2.3.1}$$

which maps a curve $x(t)$ to a number, can be expanded in a Taylor series

$$\begin{aligned} S[x(t) + \delta x(t)] &= \int_{t_\mathrm{i}}^{t_\mathrm{f}} \left\{ L + \frac{\partial L}{\partial x^\mathbf{a}} \delta x^\mathbf{a} + \frac{\partial L}{\partial \dot{x}^\mathbf{a}} \delta \dot{x}^\mathbf{a} + \mathcal{O}\left(\delta x^2\right) \right\} dt & (2.3.2) \\ &= \int_{t_\mathrm{i}}^{t_\mathrm{f}} \left\{ L + \left[\frac{\partial L}{\partial x^\mathbf{a}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^\mathbf{a}} \right) \right] \delta x^\mathbf{a} + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^\mathbf{a}} \delta x^\mathbf{a} \right) + \mathcal{O}\left(\delta x^2\right) \right\} dt & (2.3.3) \end{aligned}$$

For fixed boundary conditions, $\delta x(t_\mathrm{i}) = \delta x(t_\mathrm{f}) = 0$, the last term vanishes, leaving

$$S[x(t) + \delta x(t)] = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \left\{ L + \left[\frac{\partial L}{\partial x^\mathbf{a}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^\mathbf{a}} \right) \right] \delta x^\mathbf{a} + \mathcal{O}\left(\delta x^2\right) \right\} dt \tag{2.3.4}$$

Thus the covariant functional derivative of the action is

$$\frac{\delta S}{\delta x^\mathbf{a}} = \frac{\partial L}{\partial x^\mathbf{a}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^\mathbf{a}} \right) \tag{2.3.5}$$

and an extremum of the action is given by the **Euler-Lagrange equation**

$$\frac{\partial L}{\partial x^\mathbf{a}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^\mathbf{a}} \right) = 0 \tag{2.3.6}$$

We can also take the functional derivative with respect to the coordinate paths $x^\alpha(t)$ to get the coordinate form of the Euler-Lagrange equation

$$\frac{\delta S}{\delta x^\alpha} = \frac{\partial L}{\partial x^\alpha} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^\alpha} \right) = 0 \tag{2.3.7}$$

The action for a scalar field $\phi(x)$ has the form

$$S[\phi(x)] = \int_{t_\mathrm{i}}^{t_\mathrm{f}} L(\phi, \nabla\phi, x)\, \boldsymbol{\epsilon} \tag{2.3.8}$$

where $\boldsymbol{\epsilon}$ is the spacetime volume form. The covariant Euler-Lagrange equation is

$$\frac{\delta S}{\delta \phi} = \frac{\partial L}{\partial \phi} - \nabla_\mathbf{a} \left[\frac{\partial L}{\partial (\nabla_\mathbf{a} \phi)} \right] = 0 \tag{2.3.9}$$

where we have used the fact that the volume form is covariantly constant. We can also express the action in terms of coordinates

$$S[\phi(x)] = \int_{t_\mathrm{i}}^{t_\mathrm{f}} L(\phi, \partial\phi, x) \sqrt{|g|}\, d^4x \tag{2.3.10}$$

The coordinate Euler-Lagrange equation is then expressed in terms of the Lagrangian density $\mathcal{L} = \sqrt{|g|}\, L$ since the components of the volume form depend on the coordinates

$$\frac{\delta S}{\delta \phi} = \frac{1}{\sqrt{|g|}} \left\{ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\alpha \left[\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi)} \right] \right\} = 0 \tag{2.3.11}$$

2.3.2 Conservation laws

Eq. (2.3.6) gives the momentum conservation equation

$$\frac{dp_{\mathbf{a}}}{dt} = \frac{\partial L}{\partial x^{\mathbf{a}}} \tag{2.3.12}$$

with **momentum**

$$p_{\mathbf{a}} = \frac{\partial L}{\partial \dot{x}^{\mathbf{a}}} \tag{2.3.13}$$

which shows that the momentum is conserved if the Lagrangian is independent of x. Multiplying Eq. (2.3.6) by $\dot{x}^{\mathbf{a}}$ we get the energy conservation equation

$$\frac{dE}{dt} = -\frac{\partial L}{\partial t} \tag{2.3.14}$$

with **energy**

$$E = \dot{x}^{\mathbf{a}} \frac{\partial L}{\partial \dot{x}^{\mathbf{a}}} - L \tag{2.3.15}$$

which shows that the energy is conserved if the Lagrangian is independent of t.

Similarly, Eq. (2.3.9) gives the continuity equation

$$\nabla_{\mathbf{a}} j^{\mathbf{a}} = \frac{\partial L}{\partial \phi} \tag{2.3.16}$$

with field-space momentum[1] current

$$j^{\mathbf{a}} = \frac{\partial L}{\partial(\nabla_{\mathbf{a}} \phi)} \tag{2.3.17}$$

which shows that the field-space momentum is conserved if the Lagrangian is independent of ϕ. Multiplying Eq. (2.3.9) by $\nabla_{\mathbf{b}}\phi$ we get the energy-momentum conservation equation

$$\nabla_{\mathbf{a}} T^{\mathbf{a}}{}_{\mathbf{b}} = -\frac{\partial L}{\partial x^{\mathbf{b}}} \tag{2.3.18}$$

with **stress**(-energy-momentum) **tensor**

$$T^{\mathbf{a}}{}_{\mathbf{b}} = \frac{\partial L}{\partial(\nabla_{\mathbf{a}} \phi)} \nabla_{\mathbf{b}} \phi - L \delta^{\mathbf{a}}_{\mathbf{b}} \tag{2.3.19}$$

[1] Field-space momentum should not be confused with spacetime momentum. From the spacetime point of view, field-space momentum is a charge.

2.3.3 Symmetries and the Lie derivative

A continuous symmetry is described by the flow generated by a vector field. The **Lie derivative**, with respect to a vector field $u^{\mathbf{a}}$, acting on a vector field $v^{\mathbf{a}}$, is

$$\mathcal{L}_u v^{\mathbf{a}} = u^{\mathbf{b}}\nabla_{\mathbf{b}} v^{\mathbf{a}} - v^{\mathbf{b}}\nabla_{\mathbf{b}} u^{\mathbf{a}} \tag{2.3.20}$$

It is the derivative relative to the flow generated by $u^{\mathbf{a}}$, see Figure 2.3.1. Note that $\mathcal{L}_u$

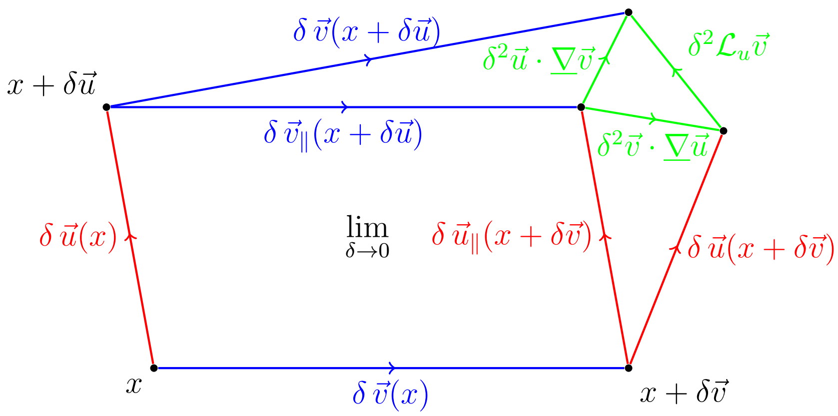

Figure 2.3.1: The Lie derivative and its relation to the covariant derivative. $\vec{v}_\parallel$ is $\vec{v}(x)$ parallel transported along $\vec{u}$, i.e. transported such that $\vec{u}\cdot\underline{\nabla}\vec{v}_\parallel = 0$, and $\vec{u}_\parallel$ is $\vec{u}(x)$ parallel transported along $\vec{v}$.

depends on $u^{\mathbf{a}}$ and its derivative, but is independent of the metric.

If a vector field $\xi^{\mathbf{a}}$ satisfies **Killing's equation**

$$\mathcal{L}_\xi g_{\mathbf{ab}} = \nabla_{\mathbf{a}}\xi_{\mathbf{b}} + \nabla_{\mathbf{b}}\xi_{\mathbf{a}} = 0 \tag{2.3.21}$$

then $\xi^{\mathbf{a}}$ is a **Killing vector field** and generates an isometry of the space.

Eqs. (2.3.12) and (2.3.13) give

$$\begin{aligned}
\frac{d}{dt}\left(\xi^{\mathbf{a}}p_{\mathbf{a}}\right) &= \xi^{\mathbf{a}}\frac{\partial L}{\partial x^{\mathbf{a}}} + \dot{\xi}^{\mathbf{a}}\frac{\partial L}{\partial \dot{x}^{\mathbf{a}}} && (2.3.22)\\
&= \xi^{\mathbf{a}}\nabla_{\mathbf{a}}L - \left(\xi^{\mathbf{b}}\nabla_{\mathbf{b}}\dot{x}^{\mathbf{a}} - \dot{\xi}^{\mathbf{a}}\right)\frac{\partial L}{\partial \dot{x}^{\mathbf{a}}} && (2.3.23)\\
&= \mathcal{L}_\xi L - \left(\mathcal{L}_\xi \dot{x}^{\mathbf{a}}\right)\frac{\partial L}{\partial \dot{x}^{\mathbf{a}}} && (2.3.24)\\
&= \mathcal{L}_\xi\big|_{\dot{x}}\, L && (2.3.25)
\end{aligned}$$

where $\mathcal{L}_\xi|_{\dot{x}}$ is the partial Lie derivative at fixed $\dot{x}^{\mathbf{a}}$. Thus $\xi^{\mathbf{a}}p_{\mathbf{a}}$ is conserved if $\xi^{\mathbf{a}}$ generates a symmetry of L. If we choose coordinates such that $e_\alpha^{\mathbf{a}} = \xi^{\mathbf{a}}$ then $\xi^{\mathbf{a}}p_{\mathbf{a}} = p_\alpha$ and its conservation can be seen directly from Eq. (2.3.7).

For example, if

$$L = \frac{1}{2}m g_{\mathbf{ab}}\dot{x}^{\mathbf{a}}\dot{x}^{\mathbf{b}} - V(x) \tag{2.3.26}$$

then

$$p_{\mathbf{a}} = mg_{\mathbf{ab}}\dot{x}^{\mathbf{b}} \tag{2.3.27}$$

and

$$\mathcal{L}_\xi|_{\dot{x}}\, L = \frac{1}{2} m\dot{x}^{\mathbf{a}}\dot{x}^{\mathbf{b}}\mathcal{L}_\xi g_{\mathbf{ab}} - \mathcal{L}_\xi V \tag{2.3.28}$$

If L has a translational symmetry generated by $e_x^{\mathbf{a}}$ then

$$e_x^{\mathbf{a}} p_{\mathbf{a}} = p_x = mg_{xx}\dot{x} = m\dot{x} \tag{2.3.29}$$

is conserved, while if L has a rotational symmetry generated by $e_\theta^{\mathbf{a}}$ then

$$e_\theta^{\mathbf{a}} p_{\mathbf{a}} = p_\theta = mg_{\theta\theta}\dot{\theta} = mr^2\dot{\theta} \tag{2.3.30}$$

is conserved.

2.3.4 Actions

Particles in spacetime

A **particle** is something that exists as a **worldline** in spacetime.

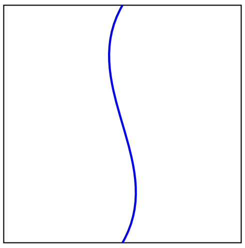

Figure 2.3.2: A particle in spacetime.

A worldline C in a spacetime M has action

$$-S[C] = \int_C (m\underline{\sigma} + q\underline{A}) \tag{2.3.31}$$

where the worldline volume form $\underline{\sigma}$ measures the length along the curve

$$d\tau = \underline{\sigma} \cdot \vec{dx} \tag{2.3.32}$$

and $\underline{A}$ is a covector field in the spacetime. Note that the physics given by $\delta S = 0$ is invariant under

$$\underline{A} \to \underline{A} + \underline{\nabla} \wedge \lambda \tag{2.3.33}$$

since

$$\int_C \underline{\nabla} \wedge \lambda = \int_{\partial C} \lambda \tag{2.3.34}$$

is a boundary term. In Lagrangian form

$$-S = \int_C \left(m\sigma_{\mathbf{a}} + qA_{\mathbf{a}}\right) dx^{\mathbf{a}} \tag{2.3.35}$$

$$= \int_C \left(m\sqrt{g_{\mathbf{ab}}\dot{x}^{\mathbf{a}}\dot{x}^{\mathbf{b}}} + qA_{\mathbf{a}}\dot{x}^{\mathbf{a}}\right) dt \tag{2.3.36}$$

The Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^{\mathbf{a}}}\right) = \frac{\partial L}{\partial x^{\mathbf{a}}} \tag{2.3.37}$$

gives

$$\frac{d}{dt}\left(p_{\mathbf{a}} + qA_{\mathbf{a}}\right) = q\left(\nabla_{\mathbf{a}} A_{\mathbf{b}}\right)\frac{dx^{\mathbf{b}}}{dt} \tag{2.3.38}$$

where the particle's momentum [2]

$$p_{\mathbf{a}} = \frac{m g_{\mathbf{ab}}}{\sqrt{g_{\mathbf{cd}}\dot{x}^{\mathbf{c}}\dot{x}^{\mathbf{d}}}}\frac{dx^{\mathbf{b}}}{dt} = m g_{\mathbf{ab}}\frac{dx^{\mathbf{b}}}{d\tau} \tag{2.3.39}$$

Therefore the force on the particle is

$$f_{\mathbf{a}} = \frac{dp_{\mathbf{a}}}{d\tau} = m g_{\mathbf{ab}}\frac{d^2x^{\mathbf{b}}}{d\tau^2} = qF_{\mathbf{ab}}\frac{dx^{\mathbf{b}}}{d\tau} \tag{2.3.40}$$

where the electromagnetic field

$$F_{\mathbf{ab}} = \nabla_{\mathbf{a}} A_{\mathbf{b}} - \nabla_{\mathbf{b}} A_{\mathbf{a}} \tag{2.3.41}$$

Eq. (2.3.40) is the relativistic form of the Lorentz force law, see Eq. (1.3.58).

[2] Note that $p_{\mathbf{a}} = m\sigma_{\mathbf{a}}$.